PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Electrification of Two Parallel Circular Discs

J. W. Nicholson

Phil. Trans. R. Soc. Lond. A 1924 **224**, 303-369
doi: 10.1098/rsta.1924.0008

VIII. *The Electrification of Two Parallel Circular Discs.*

By J. W. Nicholson, *M.A., D.Sc., F.R.S., Fellow of Balliol College, Oxford.*

Received April 24, 1924—Read June 26, 1924.

Contents.

Part I.

 [Published November 13, 1924.

Part I.

§ 1. *Introductory.*

The only problem relating to two electrified circular discs, placed parallel to each other, for which an exact solution has been obtained hitherto, is the classical one of Nobili's rings. This was solved by Riemann,* by an application of the Bessel-Fourier integral method. In this problem the discs are circular electrodes fixed to two infinite conducting planes, which are themselves connected together by the earth or by a wire at infinity. If the axis of z is that of the two co-axial discs, and perpendicular to the infinite plane conducting sheets, the electrical potential V satisfies Laplace's equation at all points between the plates, and the further conditions

$$(1)\quad \frac{\partial V}{\partial z} = 0, \qquad z = \pm a, \qquad \rho > \rho_1$$

$$(2)\quad \frac{\partial V}{\partial z} = \frac{A}{\sqrt{(r_1^2 - r^2)}}, \qquad z = \pm a, \qquad \rho < \rho_1$$

where A is a constant, $2a$ is the distance between the plates, bisected by the origin, ρ_1 is the radius of either disc, and ρ is the distance of any point from the axis of z. In fact (z, ρ) are the two cylindrical polar co-ordinates on which V can alone depend.

By pre-supposing the existence of a form for V of the type

$$V = \int_0^\infty \{\phi(\lambda)\, e^{\lambda z} + \psi(\lambda)\, e^{-\lambda z}\}\, J_0(\lambda\rho)\, d\lambda$$

clearly satisfying Laplace's equation when z is between $\pm a$, Riemann was able to determine ϕ and ψ in such a way as to make the derivate $\partial V/\partial z$ take the prescribed values all over the plane $z = a$, and thus automatically over $z = -a$ also. For the Bessel-Fourier theorem gives at once an integral of the form

$$\frac{\partial V}{\partial z} = \int_0^\infty F(\lambda)\, J_0(\lambda\rho)\, d\lambda$$

for $\partial V/\partial z$ over any plane on which its value is completely known.

A subsequent paper by Weber† and an earlier one by Kirchhoff,‡ dealing with a single circular disc, follow what is essentially a similar analysis, whose complete field of application to physical problems has never been fully worked out.

The characteristic feature of the group of problems hitherto treated in this manner is that a certain function—sometimes V and sometimes its normal derivate—is completely prescribed over an infinite plane defined by a constant value of the co-ordinate z.

* Werke, p. 58; 'Pogg. Ann.,' vol. 95, March, 1855.

† 'Crelle,' vol. 76, 1873.

‡ 'Pogg. Ann.,' vol. 64, 1845.

The next stage of complexity in such problems occurs when neither V nor its normal derivate is specified over the whole plane, but V is specified over a circular region on the plane, and $\partial V/\partial z$ over the remainder. This occurs, for example, in the problem of a single circular disc freely charged with electricity, where V is constant over the disc, and $\partial V/\partial z$ is zero over the rest of the plane. But we have no knowledge of V on the rest of the plane or of $\partial V/\partial z$ over the disc, which determines the surface density of the distribution—the object of our problem.

The problem just mentioned is analytically identical with that of the velocity potential of irrotational liquid flow through a circular aperture in an infinite plane screen, for if ϕ is the velocity potential of the motion, $\partial \phi/\partial z$ is zero over the screen, and ϕ is constant on the aperture.

For problems of this type, no direct solution by the Bessel-Fourier method has yet been found. Nevertheless, such problems often admit peculiarly neat solutions, found indirectly, in this form. The usual procedure has been to use a form of harmonic analysis, and then transform the solution into an integral form. Thus, if a circular disc is freely charged with charge Q, and occupies the plane $z = 0$ from $\rho = 0$ to $\rho = a$, we know that the surface density is

$$\sigma = -\frac{1}{4\pi}\frac{\partial V}{\partial n} = \frac{Q}{4\pi a}(a^2 - \rho^2)^{-\frac{1}{2}}$$

and $\partial V/\partial n$ is zero outside. Thus by the Bessel-Fourier theorem

$$-\frac{\partial V}{\partial n} = \frac{Q}{a}\int_0^\infty \lambda \, J_0(\lambda\rho)\, d\lambda \int_0^a \frac{\mu J_0(\lambda\mu)}{\sqrt{(a^2-\mu^2)}}\, d\mu$$
$$= \frac{Q}{a}\int_0^\infty \lambda \, J_0(\lambda\rho)\, d\lambda \, . \, \frac{\sin \lambda a}{\lambda}.$$

When this is generalised on the positive side of the axis of z,

$$-\frac{\partial V}{\partial z} = \frac{Q}{a}\int_0^\infty e^{-\lambda z} J_0(\lambda\rho) \sin \lambda a \, d\lambda$$

and

$$V = \frac{Q}{a}\int_0^\infty e^{-\lambda z} J_0(\lambda\rho) \frac{\sin \lambda a}{\lambda}\, d\lambda$$

with the resulting capacity of the disc as

$$(Q/V)_{z=0} = a\Big/\int_0^\infty J_0(\lambda\rho)\frac{\sin\lambda a}{\lambda}\, d\lambda \qquad (\rho < a)$$
$$= \frac{2a}{\pi}.$$

Similar solutions can be found at once for other physical problems relating to the circular disc. For example, in the case of a circular magnetic shell of radius a and strength ω, magnetised parallel to its axis, the magnetic potential Ω has the properties

$$\Omega = 2\pi\omega, \qquad 0 < \rho < a,$$

$$= 0, \qquad \infty > \rho > a.$$

Thus

$$\Omega = \int_0^\infty \lambda J_0(\lambda\rho)\, e^{-\lambda v}\, d\lambda \int_0^\infty \mu J_0(\lambda\mu) \,.\, 2\pi\omega\, d\mu$$

$$= 2\pi a\omega \int_0^\infty e^{-\lambda z} J_0(\lambda\rho)\, J_1(\lambda a)\, d\lambda$$

on the positive side of the z-axis.

A uniform gravitating disc of mass M also admits a similar expression, most readily found as follows:—

The potential V at an external point on the axis of z is

$$V = \frac{2M}{a^2}\left(\sqrt{(a^2+z^2)} - z\right).$$

But

$$\int_0^\infty e^{-\lambda z} \mu J_0(\lambda\mu)\, d\lambda = \frac{\mu}{\sqrt{(z^2+\mu^2)}}.$$

Integrating with respect to μ from zero to a,

$$a\int_0^\infty e^{-\lambda z} J_1(\lambda a)\frac{d\lambda}{\lambda} = \sqrt{(z^2+a^2)} - z$$

and therefore on the axis, with z positive,

$$V = \frac{2M}{a}\int_0^\infty e^{-\lambda z} J_1(\lambda a)\frac{d\lambda}{\lambda}$$

and at *any* point,

$$V = \frac{2M}{a}\int_0^\infty e^{-\lambda z} J_0(\lambda\rho)\, J_1(\lambda a)\frac{d\lambda}{\lambda}$$

the alternative expressions being elliptic integrals or series of zonal harmonics.

It does not appear to have been noticed, in regard to solutions of such problems after the present manner, that when the integrand consists of the product of $e^{-\lambda z}J_0(\lambda\rho)$ and another Bessel function—sin λa is effectively a Bessel function of order $\frac{1}{2}$—that function is always of integer order for any rigid distribution of the system, for example, a rigid distribution of magnetic doublets, as in a circular shell, or of attracting matter in the last problem. On the other hand, in *free* distributions such as are found in electricity or hydrodynamics, where the electricity or the fluid is free to move on or about the surface, the order is always half an odd integer. The freely charged circular disc is one

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

instance. Another is the expression for the velocity potential ϕ created by a disc moving broadside on in incompressible liquid, which is given by*

$$\phi = -\frac{2V}{\pi}\int_0^\infty e^{-\lambda z} J_0(\lambda\rho)\,\frac{d}{d\lambda}\frac{\sin\lambda a}{\lambda}\,d\lambda$$

where V is the velocity of the disc, and $d/d\lambda\,(\sin\lambda a/\lambda)$ is effectively $J_{3/2}(a\lambda)$.

Some general remarks concerning the nature of the difficulties which arise, in the attempt to find *direct* solutions of other problems by the method of discontinuous integrals, are of interest at this point. The essentials of the direct problem may be realised by taking a simple case, such as the specification

$$\phi = F(\rho) \qquad \infty > \rho > a \qquad z = 0$$

$$\frac{\partial\phi}{\partial z} = f(\rho) \qquad 0 < \rho < 0 \qquad z = 0$$

where ϕ is to be a solution of LAPLACE's equation.

Let the unknown value of ϕ when $0 < \rho < a$ on the plane be $f_1(\rho)$.

Then by the Bessel-Fourier theorem, the complete specification of ϕ on the plane is

$$\phi = \int_0^\infty \lambda J_0(\lambda\rho)\,d\lambda \int_0^a f_1(\mu)\,J_0(\lambda\mu)\,\mu\,d\mu + \int_0^\infty \lambda J_0(\lambda\rho)\,d\lambda \int_a^\infty F(\mu)\,J_0(\lambda\mu)\,\mu\,d\mu,$$

and thus for all points with z positive,

$$\phi = \int_0^\infty \lambda e^{-\lambda z} J_0(\lambda\rho)\,d\lambda \left\{\int_0^a f_1(\mu) + \int_a^\infty F(\mu)\right\}\mu J_0(\lambda\mu)\,d\mu,$$

and

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=0} = -\int_0^\infty \lambda^2 J_0(\lambda\rho)\,d\lambda\left\{\int_0^a f_1(\mu) + \int_a^\infty F(\mu)\right\}\mu J_0(\lambda\mu)\,d\mu,$$

where f_1 is unknown. Let now the value of $(\partial\phi/\partial z)_{z=0}$ be $f_2(\rho)$ when ρ is greater than a. This is also unknown.

Then directly we have also,

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=0} = \int_0^\infty \lambda J_0(\lambda\rho)\,d\lambda\left\{\int_0^a f(\mu) + \int_a^\infty f_2(\mu)\right\}\mu J_0(\lambda\mu)\,d\mu.$$

These expressions must be identical. To determine the two unknowns (f_1, f_2) we have no further information beyond the fact that $f_1(\rho)$ and $f_2(\rho)$ are particular cases of two solutions of LAPLACE's equation with z equated to zero—and usually a known type of behaviour of ϕ at infinity.

The mere identity of the two expressions could be secured by taking

$$\int_0^a \{f(\mu) + \lambda f_1(\mu)\}\,\mu J_0(\lambda\mu)\,d\mu = -\int_a^\infty \{f_2(\mu) + \lambda F(\mu)\}\mu J_0(\lambda\mu)\,d\mu,$$

* LAMB, 'Hydrodynamics,' Camb. Univ. Press.

but an infinite number of pairs of functions can in general be found with this property, and there is no mode of selection of the pair with the necessary relation to LAPLACE'S equation, if such a pair exists. In cases which we have tried, no such pair does exist, and in taking the last step, we have lost the solution which is sought, and which appears to be beyond the power of the theory of integral equations in its present state.

These remarks illustrate the bearing of the problem on its direct side, and will be of use later. They serve to indicate the nature of the limitations to the method of discontinuous integrals, whose solutions are of great value from the facility with which the solutions on the plane $z = 0$ can be generalised at once by introduction of the factor $e^{-\lambda z}$. These solutions on the plane must, however, be found otherwise in the first place, and have been found usually by a process of harmonic analysis.

In the present memoir, we take up the problem of two parallel electrified discs, on the same axis, as the primary theme. In the course of its solution, other problems are also solved. It is clear that this problem presents the next stage in difficulty, by the method of discontinuous integrals, after those just mentioned. For the conditions are that the potential V is constant on each disc, but we have no knowledge either of V or $\partial V/\partial n$ on the rest of the plane of either disc.

The mode of solution is, initially, by a harmonic analysis which treats the discs as special cases of oblate spheroids. The necessary analysis preliminary to the problem has been developed in an earlier memoir* and need not be repeated. We shall refer continually to this memoir as 'O.S.H.' The harmonic analysis itself does not effect the solution, but it leads us to an integral equation of a new type, which is ultimately solved. The value of the potential can then be expressed in a variety of ways, including a Bessel-Fourier integral form.

§ 2. *Transformation of Spheroidal Harmonics to a New Origin.*

Some very interesting formulæ which transform products of spheroidal harmonics to harmonic series about another origin do not appear to have been noticed. We shall only include, in this memoir, one which is especially fundamental for problems of the type contemplated. Corresponding formulæ of other types may be found after the same manner.

If O_1 and O_2 are two origins on the axis of z, O_1 being at $z = -c$ on the left, and O_2 at $z = 0$, any point ρ of space, in a problem symmetrical about the axis of z, may be defined by its cylindrical co-ordinates (z, ρ), where ρ is the perpendicular from ρ on the z-axis, or by its spheroidal (oblate) co-ordinates with respect to the origins O_2 and O_1. Let these be, respectively, (μ, ζ) and (μ', ζ'), the accent thus relating to the origin $z = -c$. Then

$$z = a\mu\zeta, \qquad \rho = a\sqrt{\{(1-\mu^2)(1+\zeta^2)\}}$$
$$z + c = a\mu'\zeta', \qquad \rho = a\sqrt{\{(1-\mu'^2)(1+\zeta'^2)\}}$$

* "Oblate Spheroidal Harmonics," 'Phil. Trans.' A, vol. 224, pp. 49–93.

the typical spheroids (ζ, ζ') constant, of the two confocal systems being respectively

$$\frac{z^2}{a^2\zeta^2}+\frac{\rho^2}{a^2(1+\zeta^2)}=1, \qquad \frac{(z+c)^2}{a^2\zeta'^2}+\frac{\rho^2}{a^2(1+\zeta'^2)}=1.$$

Accordingly, the co-ordinates are related by

$$\mu'\zeta'=\frac{c}{a}+\mu\zeta,$$
$$(1-\mu'^2)(1+\zeta'^2)=(1-\mu^2)(1+\zeta^2)$$

from which (μ', ζ') relating to O_1 on the left, could be found. These relations are, however, in themselves sufficient for our purpose.

Now the function

$$P_n(\mu')\,q_n(\zeta')$$

is a solution of LAPLACE'S equation, and our object is to expand it into a series of the form

$$P_n(\mu')\,q_n(\zeta')=\sum_0^\infty P_r(\mu)\,p_r(\zeta)\,f_r(c/a),$$

which must evidently exist, for a suitable range of values of the variables. If this can be effected, we can express a potential function diverging from O_1 in a form suitable for a function converging on O_2.

As a preliminary, we desire an expression of $P_n(\mu')$, $q_n(\zeta')$ in the form of a definite integral. In the theory of spherical harmonics, there is a well-known formula

$$\int_0^\pi Q_n\{\mu\zeta+\sqrt{\{(1-\mu^2)(1-\zeta^2)\}}\cos\phi\}\,d\phi=\pi P_n(\mu)\,Q_n(\zeta),$$

where $\zeta > \mu$. Its analogue in terms of q-functions is readily shown to be

$$\pi P_n(\mu)\,q_n(\zeta)=\int_0^\pi q_n\{\mu\zeta-\iota\sqrt{\{(1-\mu^2)(1+\zeta^2)\}}\cos\phi\}\,d\phi, \quad . \quad . \quad . \quad (1)$$

the right-hand side being only apparently complex. In this formula ζ is not restricted, when it is real. The proof can easily be supplied by the reader.

Thus,

$$\pi P_n(\mu')\,q_n(\zeta')=\int_0^\pi q_n\{\mu'\zeta'-\iota\sqrt{\{(1-\mu'^2)(1+\zeta'^2\}}\cos\phi\}\,d\phi,$$

or, in terms of the new co-ordinates (μ, ζ) for the origin O_2,

$$\pi P_n(\mu')\,q_n(\zeta')=\int_0^\pi q_n\left\{\frac{c}{a}+\mu\zeta-\iota\sqrt{\{1-\mu^2)(1+\zeta^2)\}}\cos\phi\right\}d\phi.$$

We shall return to this formula later. In particular, with $n = 0$.

$$\pi q_0(\zeta') = \int_0^\pi q_0\left(\frac{c}{a} + \varepsilon\right) d\phi$$

where

$$\varepsilon = \mu\zeta - \iota\sqrt{\{(1-\mu^2)(1+\zeta^2)\}} \cos\phi.$$

Using the series development for q_0 in the integrand—it is convergent absolutely and uniformly at least when $c > a$, and actually under yet closer restrictions—and writing $c/a = \lambda$,

$$\pi q_n(\zeta') = \int_0^\pi d\phi \left\{\frac{1}{\lambda+\varepsilon} - \frac{1.2}{2.3}\frac{1}{(\lambda+\varepsilon)^3} + \frac{1.2.3.4}{2.4.3.5}\frac{1}{(\lambda+\varepsilon)^5} - \ldots\right\}$$

$$= \left(1 - \frac{1.2}{2.4}\cdot\frac{1}{2!}\frac{\partial^2}{\partial\lambda^2} + \frac{1.2.3.4}{2.4.3.5}\frac{1}{4!}\frac{\partial^4}{\partial\lambda^4} - \ldots\right)\int_0^\pi \frac{d\phi}{\lambda+\varepsilon}$$

$$= \frac{\sin\partial/\partial\lambda}{\partial/\partial\lambda}\cdot\int_0^\pi \frac{d\phi}{\lambda+\varepsilon}.$$

Thus if

$$D \equiv \partial/\partial(c/a) = \partial/\partial\lambda,$$

$$q_0(\zeta') = \frac{\sin D}{\pi D}\int_0^\pi d\phi \Big/ \left\{\lambda + \mu\zeta - \iota\sqrt{\{(1-\mu^2)(1+\zeta^2)\}}\cos\phi\right\} \quad . \quad . \quad . \quad (2)$$

The integral involved here must evidently itself be a solution of LAPLACE's equation, as can readily be verified. Its value, in one form, is easily shown to be $\pi a/O_1P$ or $\pi a/\sqrt{\{\rho^2 + z^2\}}$. Thus, if ζ' is related to an origin O_2 at a distance c behind O_1 on the z-axis,

$$q_0(\zeta') = \frac{\sin D}{\pi D}\cdot\frac{\pi a}{O_1 P} = \left(\sin a\frac{\partial}{\partial c}\right)\Big/\left(\frac{a\partial}{\partial c}\right)\cdot\frac{a}{O_1 P}$$

where a^2 is the difference of squared semiaxes of the primary confocal spheroids, $\zeta = 0$ and $\zeta' = 0$.

Now, by the "inverse distance formula" of 'O.S.H.,'* namely,

$$\frac{a}{O_1 P} = \sum_0^\infty (-)^r (2r+1) P_r(\mu) p_r(\zeta) q_r\left(\frac{c}{a}\right)$$

we at once deduce

$$q_0(\zeta') = \frac{\sin D}{D}\left\{q_0\left(\frac{c}{a}\right) - 3P_1(\mu) p_1(\zeta) q_1\left(\frac{c}{a}\right) + 5P_2(\mu) p_2(\zeta) q_2\left(\frac{c}{a}\right) - \ldots\right\}$$

when c/a is greater than ζ, as will occur in the applications contemplated, where the formula is only employed in the immediate neighbourhood of the surface of a second spheroid equal to the first, which it does not intersect. Thus

$$q_0(\zeta') = \sum_0^\infty (-)^r (2r+1) P_r(\mu) p_r(\zeta)\cdot\frac{\sin D}{D} q_r\left(\frac{c}{a}\right). \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* *Loc. cit.*, p. 54.

After the same manner we can proceed to the more general formula, since, for sufficiently large values of

$$\frac{c}{a} + \varepsilon \equiv \frac{c}{a} + \mu\zeta - \iota\sqrt{\{(1-\mu^2)(1+\zeta^2)\}}\cos\phi,$$

if $c/a = \lambda$, we have

$$q_n\left(\frac{c}{a}+\varepsilon\right) = \frac{2^n (n\,!)^2}{2n+1\,!}\left\{\frac{1}{(\lambda+\varepsilon)^{n+1}} - \frac{(n+1)(n+2)}{2(2n+3)}\cdot\frac{1}{(\lambda+\varepsilon)^{n+3}} + \ldots\right\}.$$

Accordingly

$$P_n(\mu')\, q_n(\zeta') = \int_2^\pi q_n\left(\frac{c}{a}+\varepsilon\right) d\phi$$

where

$$\varepsilon = \mu\zeta - c\sqrt{\{(1-\mu^2)(1+\zeta^2)\}}\cos\phi,$$

or

$$P_n(\mu')\, q_n(\zeta') = \frac{2^n (n\,!)^2}{(2n+1)\,!}\int_0^\pi d\phi\left\{\frac{1}{(\lambda+\varepsilon)^{n+1}} - \frac{(n+1)(n+2)}{2(2n+3)}\frac{1}{(\lambda+\varepsilon)^{n+3}} + \ldots\right\}$$

$$= (-)^n \frac{2^n n\,!}{(2n+1)\,!} D^n \left\{1 - \frac{D^2}{2\,.\,(2n+3)} + \frac{D^4}{2\,.\,4\,.\,(2n+3)(2n+5)} + \ldots\right\}\int_0^\pi \frac{d\phi}{\lambda+\varepsilon}$$

$$= \frac{(-2)^n n\,!}{(2n+1)\,!}\frac{J_{n+\frac{1}{2}}(D)}{\sqrt{D}}\,.\,2^{n+\frac{1}{2}}\Gamma\left(n+\tfrac{3}{2}\right)\cdot\frac{a\pi}{O_1P}$$

$$= (-)^n\left(\frac{\pi}{2D}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(D)\,\frac{a\pi}{O_1P}$$

in the preceding notation. Finally, with the aid of the inverse distance formula,

$$P_n(\mu')\, q_n(\zeta') = (-)^n\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_0^\infty (-)^r (2r+1)\, P_r(\mu)\, p_r(\zeta)\,.\,\frac{J_{n+\frac{1}{2}}(D)}{\sqrt{D}}\, q_r\left(\frac{c}{a}\right) \quad . \quad (4)$$

where $D \equiv \partial/\partial(c/a)$.

We are accordingly led to a consideration of the function $K_r^n(x)$ defined by

$$\frac{\pi}{2} K_r^n(x) = (-)^n\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{J_{n+\frac{1}{2}}(\partial/\partial x)}{\sqrt{(\partial/\partial x)}}\, q_r(x) \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

and in terms of this function

$$P_n(\mu')\, q_n(\zeta') = \frac{\pi}{2}\sum_0^\infty (-)^r (2r+1)\, P_r(\mu)\, p_r(\zeta)\, K_r^n\left(\frac{c}{a}\right) \quad . \; . \; . \; . \quad (6)$$

which is the required formula of transformation.

§ 3. *The function* $K_r^n(x)$.

By a formula of 'O.S.H.'* $q_r^n(x)$ is given by

$$q_r(x) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_0^\infty e^{-\lambda x} J_{r+\frac{1}{2}}(\lambda) \frac{d\lambda}{\sqrt{\lambda}}$$

so that

$$K_r^n(x) = (-)^n \frac{J_{n+\frac{1}{2}}(\partial/\partial x)}{\sqrt{(\partial/\partial x)}} \int_0^\infty e^{-\lambda x} J_{r+\frac{1}{2}}(\lambda) \frac{d\lambda}{\sqrt{\lambda}}$$

or

$$K_r^n(x) = \int_0^\infty e^{-\lambda x} J_{r+\frac{1}{2}}(\lambda) J_{n+\frac{1}{2}}(\lambda) \frac{d\lambda}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

representing the function as a definite integral. This is in many ways the most convenient form of the function for our purposes. It shows, incidentally, that $K_r^n(x)$ is a function symmetrical in r and n, and thus leads to a very remarkable property of the spheroidal harmonics in the form

$$(-)^n \frac{J_{n+\frac{1}{2}}(D)}{\sqrt{D}} q_r(x) = (-)^r \frac{J_{r+\frac{1}{2}}(D)}{\sqrt{D}} q_n(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $D \equiv \partial/\partial x$.

It is of some importance to obtain a series for $K_r^n(x)$ in ascending powers of x^{-1}. This may be found most readily as follows:—

By a well-known formula†

$$J_{n+\frac{1}{2}}(\lambda) J_{r+\frac{1}{2}}(\lambda) = \frac{\Gamma(n+r+1)}{2^{n+r+1}\Gamma(n+\frac{3}{2})\Gamma(r+\frac{3}{2})} S$$

where

$$S = 1 - \frac{n+r+3}{2!(n+\frac{3}{2})(r+\frac{3}{2})}\left(\frac{\lambda}{2}\right)^2 + \frac{(n+r+4)(n+r+5)}{4!(n+\frac{3}{2})(n+\frac{5}{2})(r+\frac{3}{2}(r+\frac{5}{2})}\left(\frac{\lambda}{2}\right)^4 - \ldots$$

or in brief notation

$$J_{n+\frac{1}{2}}(\lambda) J_{r+\frac{1}{2}}(\lambda) = \sum_{s=0}^{\infty} (-)^s \left(\frac{\lambda}{2}\right)^{n+r+2s+1} \frac{(n+r+2s+1)!}{s!(n+r+s+1)!\Gamma(r+s+\frac{3}{2})\Gamma(n+s+\frac{3}{2})}$$

and since all numbers are integers,

$$\int_0^\infty \lambda^{n+r+2s} e^{-\lambda x} dx = (n+r+2s)!\, x^{-n-r-2s-1}$$

and we find

$$K_r^n(x) = \frac{n+r!}{\Gamma(n+\frac{3}{2})\Gamma(r+\frac{3}{2})} \cdot \frac{1}{(2x)^{n+r+1}} F$$

* *Loc. cit.* p. 56.

† SCHLÄFLI, 'Math. Ann.' (III), 1871, p. 141, and others.

where F is a hypergeometric function with four numerators and four denominators, of the form

$$F \equiv F\left\{\tfrac{1}{2}(n+r+1),\ \tfrac{1}{2}(n+r+2),\ \tfrac{1}{2}(n+r+2),\ \tfrac{1}{2}(n+r+3);\ n+r+2,\ n+\tfrac{3}{2},\ r+\tfrac{3}{2};\ -4/x^2\right\}. \quad (9)$$

This satisfies a linear differential equation of the fourth order.

Other definite integrals which represent the function can be obtained readily. For example, we know that

$$J_{n+\frac{1}{2}}(\lambda)\, J_{r+\frac{1}{2}}(\lambda) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} J_{n+r+1}(2\lambda\cos\phi)\cos(n-r)\phi\, d\phi$$

and therefore

$$K_r^n(x) = \int_0^\infty e^{-\lambda x} J_{n+\frac{1}{2}}(\lambda)\, J_{r+\frac{1}{2}}(\lambda)\frac{d\lambda}{\lambda}$$
$$= \int_0^{\frac{1}{2}\pi}\cos(n-r)\phi\, d\phi \int_0^\infty e^{-\lambda x} J_{n+r+1}(2\lambda\cos\phi)\frac{d\lambda}{\lambda}$$

by an obviously valid inversion of the order of integration.

Moreover,

$$2(n+r+1)\, J_{n+r+1}(2\lambda\cos\phi) = 2\lambda\cos\phi\,(J_{n+r} + J_{n+r+2})$$

and therefore

$$K_r^n(x) = \frac{2}{\pi(n+r+1)}\int_0^{\frac{1}{2}\pi}\cos\phi\cos(n-r)\phi\, d\phi\int_0^\infty d\lambda e^{-\lambda x}\{J_{n+r}(2\lambda\cos\phi) + J_{n+r+2}(2\lambda\cos\phi)\}.$$

But HEINE and PINCHERLE have independently given the formula

$$\int_0^\infty J_n(a\lambda)\, e^{-b\lambda}\, d\lambda = \frac{\{(a^2+b^2)^{\frac{1}{2}} - b\}^n}{a^n (a^2+b^2)^{\frac{1}{2}}}$$

and we deduce

$$K_r^n(x) = \frac{2}{\pi(n+r+1)}\int_0^{\frac{1}{2}\pi}\frac{\cos\phi\cos(n-r)\phi\, d\phi}{\sqrt{(x^2+4\cos^2\phi)}}\left\{\frac{\sqrt{(x^2+4\cos^2\phi)}-\lambda}{2\cos\phi}\right\}^{n+r}\left\{1+\left(\frac{\sqrt{(x^2+4\cos^2\phi)}}{2\cos\phi}\right)^2\right\}$$

or on reduction,

$$K_r^n(x) = \frac{1}{\pi(n+r+1)}\int_0^{\frac{1}{2}\pi}\cos(n-r)\phi\left(\frac{\sqrt{(x^2+4\cos^2\phi)}-x}{2\cos\phi}\right)^{n+r+1} d\phi. \quad (10)$$

We proceed to find the linear differential equation satisfied by the function.

With $\qquad -4/x^2 = y, \qquad \vartheta_1 = y\,\partial/\partial y,$

$$\vartheta = x\,\partial/\partial x = -\tfrac{1}{2}\vartheta_1, \qquad x = e^\theta, \qquad \vartheta \equiv \partial/\partial\theta.$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The equation in y may be written down by inspection, from the general differential equation for functions of hypergeometric type. Thus F satisfies

$$\{\vartheta_1 + \tfrac{1}{2}(n+r+1)\}\{\vartheta_1 + \tfrac{1}{2}(n+r+2)\}^2\{\vartheta_1 + \tfrac{1}{2}(n+r+3)\}\mathrm{F}$$
$$-\frac{1}{y}\{\vartheta_1(\vartheta_1+n+\tfrac{1}{2})(\vartheta_1+r+\tfrac{1}{2})(\vartheta_1+n+r+1)\}\mathrm{F} = 0,$$

or if $x = e^\theta, \quad x\partial/\partial x = \partial/\partial\theta = \vartheta = -\tfrac{1}{2}\vartheta_1$

$$(\vartheta-n-r-1)(\vartheta-n-r-2)^2(\vartheta-n-r-3)\mathrm{F}$$
$$+\frac{x^2}{4}\vartheta(\vartheta-2n-1)(\vartheta-2r-1)(\vartheta-2n-2r-2)\mathrm{F} = 0.$$

Now $\mathrm{K}_r^n(x)$, or more briefly K, is proportional to $x^{-n-r-1}\mathrm{F}$, and accordingly

$$(\vartheta-n-r-1)(\vartheta-n-r-2)^2(\vartheta-n-r-3)\mathrm{F}$$
$$=(\vartheta-n-r-1)(\vartheta-n-r-2)^2(\vartheta-n-r-3)\mathrm{K}e^{\theta(n+r+1)}$$
$$=e^{(n+r+1)\theta}\vartheta(\vartheta-1)^2(\vartheta-2)\mathrm{K}$$

and

$$\vartheta(\vartheta-2n-1)(\vartheta-2r-1)(\vartheta-2n-2r-2)\mathrm{F}$$
$$=e^{(n+r+1)\theta}(\vartheta+n+r+1)(\vartheta+r-n)(\vartheta+n-r)(\vartheta-n-r-1)\mathrm{K},$$

and finally

$$\vartheta(\vartheta-1)^2(\vartheta-2)\mathrm{K}+\frac{x^2}{4}\{\vartheta^2-(n+r+1)^2\}\{\vartheta^2-(n-r)^2\}\mathrm{K}=0 \quad (11)$$

is the required differential equation.

§ 4. *The value of* $\mathrm{K}_n^0(x)$.

When $r = 0$, the function is comparatively simple. Its equation is

$$\vartheta(\vartheta-1)^2(\vartheta-2)\mathrm{K}-\frac{x^2}{4}\vartheta^2\{\vartheta^2-(n+1)^2\}\mathrm{K}=0.$$

Now consider the expression

$$y = (1+\zeta^2)\frac{dq_n(\zeta)}{d\zeta}.$$

Since

$$\frac{d}{d\zeta}(1+\zeta^2)\frac{dq_n}{d\zeta} = n(n+1)q_n$$

we find at once that

$$(1+\zeta^2)\frac{d^2y}{d\zeta^2} = n(n+1)y.$$

Consider the function y of a complex argument $\zeta = x + \iota$, and let

$$y = \mathrm{A} + \iota\mathrm{B}$$

where A and B are real. In other words

$$\left[(1+\zeta^2)\frac{dq_n}{d\zeta}\right]_{\zeta=x+\iota} = \mathrm{A}+\iota\mathrm{B}.$$

We find, if an accent denotes differentiation to x,

$$(x^2+2\iota x)(\mathrm{A}''+\iota\mathrm{B}'') = n(n+1)(\mathrm{A}+\iota\mathrm{B})$$

or

$$n^2\mathrm{A}'' - n(n+1)\mathrm{A} = 2x\mathrm{B}''$$
$$x^2\mathrm{B}'' - n(n+1)\mathrm{B} = -2x\mathrm{A}''$$

which are equivalent to

$$(\vartheta-n)(\vartheta+n+1)\mathrm{A} = \frac{2}{x}\vartheta(\vartheta-1)\mathrm{B}$$

$$(\vartheta-n)(\vartheta+n+1)\mathrm{B} = -\frac{2}{x}\vartheta(\vartheta-1)\mathrm{A}$$

and eliminating B at once, we obtain the differential equation for A. The same equation is satisfied by B, and it is in fact clear that the general solution of this equation of the fourth order is

$$\mathrm{K}_n{}^0 = y = [\alpha(1+\zeta^2)q_n'(\zeta)+\beta(1+\zeta^2)p_n'(\zeta)]_{\zeta=x+\iota} + [\gamma(1+\zeta^2)q_n'(\zeta)+\delta(1+\zeta^2)p_n'(\zeta)]_{\zeta=x-\iota}. \quad (12)$$

The four fundamental solutions are in fact the real and imaginary parts of

$$(1+\zeta^2)[p_n'(\zeta), (q_n'(\zeta)]$$

when $\zeta = x+\iota$. They thus have a relation to the spheroidal harmonics analogous to that between KELVIN'S *ber* and *bei* functions and the Bessel functions.

To obtain $\mathrm{K}_n{}^0(x)$ directly, with this information, we have

$$\mathrm{K}_n{}^0(x) = \int_0^\infty e^{-\lambda x}\mathrm{J}_{n+\frac{1}{2}}(\lambda)\frac{d\lambda}{\lambda}\left(\frac{2}{\pi\lambda}\right)^{\frac{1}{2}}\sin\lambda.$$

Now

$$\int_0^\infty e^{-\lambda x}\mathrm{J}_{n+\frac{1}{2}}(\lambda)\frac{d\lambda}{\sqrt{\lambda}} = \sqrt{\frac{2}{\pi}}\cdot q_n(x)$$

whether x be entirely real or not. Thus, writing $x+\iota$ for x, we find by the usual inversion of order of integration, clearly admissible, that

$$\int_0^\infty \mathrm{J}_{n+\frac{1}{2}}(\lambda)\frac{d\lambda}{\sqrt{\lambda}}\cdot e^{-\lambda\iota}\int_x^\infty e^{-\lambda x}dx = \sqrt{\frac{2}{\pi}}\int_x^\infty q_n(x+\iota)\,dx$$
$$= -\sqrt{\frac{2}{\pi}}\frac{1+\zeta^2}{n(n+1)}q_n'(\zeta)$$

where $\zeta = x + \iota$. Similarly, if $\zeta' = x - \iota$, we obtain a similar expression, and by subtraction,

$$\int_0^\infty e^{-\lambda x} J_{n+\frac{1}{2}}(\lambda) \frac{\sin\lambda}{\lambda^{\frac{3}{2}}} d\lambda = \tfrac{1}{2}\iota^{-1}\sqrt{\frac{2}{\pi}}\left\{\frac{(1+\zeta^2)\,q_n'(\zeta) - (1+\zeta'^2)\,q_n'(\zeta')}{n(n+1)}\right\}.$$

This can be transformed by use of the recurrence formulæ

$$(1+\zeta^2)\,q_n' = -(n+1)(q_{n+1} + \zeta q_n)$$

$$0 = (n+1)\,q_{n+1} - (2n+1)\,\zeta q_n - n q_{n-1}$$

into the final result

$$K_n^0(x) = K_0^n(x) = -\frac{1}{\iota\pi(2n+1)}\{q_{n-1}(x+\iota) + q_{n+1}(x+\iota) - q_{n-1}(x-\iota) - q_{n+1}(x-\iota)\}. \quad (13)$$

The transformation formula for $q_0(\zeta')$ can now be exhibited entirely in P and q functions.

§ 5. *Further Examination of* $K_r^n(x)$.

This more general function cannot be dealt with so simply. From its importance in problems of the present nature, however, it is worthy of a detailed study which cannot be given in this memoir. We shall merely indicate other interesting properties of considerable use.

In the first place, it admits a useful recurrence formula, which can serve in problems for which not an exact solution, but only an approximate one, is needed.

Slightly generalising the function, as is necessary later, into

$$K_r^n(x) = \int_0^\infty e^{-\lambda x} J_{n+\frac{1}{2}}(\lambda)\, J_{r+\frac{1}{2}}(a\lambda)\, \frac{d\lambda}{\lambda}$$

where a is a new parameter, we have

$$K_{r+1}^n(x) = -\int_0^\infty e^{-\lambda x} J_{n+\frac{1}{2}}(\lambda) \frac{d\lambda}{\lambda^2} a^{r+\frac{1}{2}} \frac{\partial}{\partial a}\{a^{-(r+\frac{1}{2})} J_{r+\frac{1}{2}}(\lambda a)\}$$

(by the recurrence formula for Bessel functions)

$$= -\int_0^\infty e^{-\lambda x} J_{n+\frac{1}{2}}(\lambda) \frac{d\lambda}{\lambda}\left\{J'_{r+\frac{1}{2}}(\lambda a) + \frac{2r+1}{2a\lambda} J_{r+\frac{1}{2}}(\lambda a)\right\}.$$

But

$$2J'_{r+\frac{1}{2}} = J_{r-\frac{1}{2}} - J_{r+\frac{3}{2}},$$

so that we may write

$$K_{r+1}^n = -\tfrac{1}{2}(K_{r-1}^n - K_{r+1}^n) + \frac{(r+\frac{1}{2})}{a}\int_x^\infty K_r^n(x)\, dx$$

or

$$K_{r+1}^n + K_{r-1}^n = \frac{2r+1}{a}\int_x^\infty K_r^n(x)\, dx, \quad \ldots \quad (14)$$

and calculation from the first two functions K_0^n, K_{-1}^n (whose value is obtained at once) is very rapid. K_{-1}^n is the conjugate complex to K_0^n.

One further integral expression of a very different type must be indicated.

We have, if $c/a = x$,

$$\int_0^\pi q_n\{x + \mu\zeta - \iota\sqrt{\{(1-\mu^2)(1+\zeta^2)\}}\cos\phi\}\,d\phi = \pi P_n(\mu')\,q_n(\zeta'),$$

and, therefore,

$$\int_0^\pi q_n\{x + \mu\zeta - \iota\sqrt{(1-\mu^2)(1+\zeta^2)}\cos\phi\}\,d\phi$$
$$= \frac{\pi^2}{2}\sum_0^\infty(-)^r(2r+1)\,P_r(\mu)\,p_r(\zeta)\,K_r^n(x).$$

This is a very convergent series if x is greater than $|\zeta|$, and remains true if ζ is wholly imaginary. Thus

$$\int_0^\pi q_n\{x + \mu\iota\zeta - \iota\sqrt{\{1-\mu^2)(1-\zeta^2)}\cos\phi\}\,d\phi$$
$$= \frac{\pi^2}{2}\sum_0^\infty(-\iota^{-r})(2r+1)\,P_r(\mu)\,P_r(\zeta)\;K_r^n(x),$$

and in particular, with $\mu = 1$

$$\int_0^\pi q_n(x+\iota\zeta)\,d\phi = \pi q_n(x+\iota\zeta)$$
$$= \frac{\pi^2}{2}\sum_0^\infty(-\iota^{-r})(2r+1)\,P_r(\zeta)\,K_r^n(x).$$

Thus, $K_r^n(x)$ arises in the expression of $q_n(x+\iota\zeta)$ in a series of Legendre coefficients. In fact

$$K_r^n(x) = \frac{\iota^{-r}}{\pi}\int_{-1}^1 q_n(x+\iota\zeta)\,P_r(\zeta)\,d\zeta.$$

This expression is not obviously symmetrical in r and n, but can readily be proved to be so. For whether x be real or not,

$$Q_n(x) = \tfrac{1}{2}\int_{-1}^1 \frac{P_n(y)}{x-y}\,dy, \qquad q_n(x) = \tfrac{1}{2}\iota^{-(n+1)}\int_{-1}^1 \frac{P_n(y)\,dy}{\iota x - y}$$

—the principal value being taken in the first if x is a real number between ± 1. Thus

$$K_r^n(x) = \frac{\iota^{-r-n-1}}{2\pi}\int_{-1}^1 P_r(\zeta)\,d\zeta\int_{-1}^1 \frac{P_n(y)\,dy}{\iota(x+\iota\zeta)-y}$$

and inverting the order,

$$K_r^n(x) = \frac{\iota^{-r-n-1}}{2\pi}\int_{-1}^{1} P_n(y)\,dy\int_{-1}^{1}\frac{P_r(\zeta)\,d\zeta}{\iota x - y - \zeta},$$

$$= +\frac{\iota^{-r-n-1}}{2\pi}\int_{-1}^{1} P_n(y)\,dy\int_{-1}^{1}\frac{P_r(\zeta)\,d\zeta}{\iota(x+\iota y)-\zeta},$$

$$= +\frac{1}{2\pi}\iota^{-r-n-1}\int_{-1}^{1} P_n(y)\,Q_r\{\iota(x+\iota y)\}\,dy,$$

$$= +\frac{1}{\pi}\iota^{-r-n-1+r+1}\int_{-1}^{1} P_n(y)\,q_r(x+\iota y)\,dy.$$

The power of ι is $-n$, and therefore the integral is symmetrical, and we find

$$K_r^n(x) = \frac{\iota^{-r}}{\pi}\int_{-1}^{1} q_n(n+\iota\zeta)\,P_r(\zeta)\,d\zeta$$

$$= \frac{\iota^{-n}}{\pi}\int_{-1}^{1} q_r(x+\iota\zeta)\,P_n(\zeta)\,d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

both expressions being real. The general equation of transformation to new spheroidal co-ordinates can therefore be written

$$P_n(\mu')\,q_n(\zeta') = \left(\frac{\iota}{2}\right)^{-n}\sum_0^\infty (-)^r(2r+1)\,P_r(\mu)\,q_r(\zeta)\int_{-1}^{1} P_n(\lambda)\,q_r\left(\frac{c}{a}+\iota\lambda\right)d\lambda$$

$$= \tfrac{1}{2}\sum_0^\infty (-\iota)^r(2r+1)\,P_r(\mu)\,q_r(\zeta)\int_{-1}^{1} P_r(\lambda)\,q_r\left(\frac{c}{a}+\iota\lambda\right)d\lambda \quad (16)$$

in a form containing only P and q functions.

§ 6. *Two Equal and Equally Charged Coaxial Spheroids.*

Let O_1 and O_2 be the centres of two equal, non-intersecting coaxial oblate spheroids, their axes of revolution being along z, O_2 at the origin $z = 0$, and O_1 at $z = -c$, so that c is the distance between centres. Let ζ_0 be the parameter of the spheroid round O_2, related to the harmonics at O_2, and ζ_0' the similar parameter for the spheroid round O_1. Since they are equal spheroids, $\zeta_0 = \zeta_0'$ numerically. The line constant of harmonics is a. When c/a is large, the system is effectively equivalent to two non-influencing spheroids. A single charged spheroid at the origin (O_2) gives a potential proportional to $q_0(\zeta)$ at external points, so that the first approximation to the total external potential of the two spheroids, when equally charged, is

$$V = q_0(\zeta) + q_0(\zeta')$$

(the actual charge being omitted for convenience).

The general exact expression must be of the type

$$V = q_0(\zeta) + q_0(\zeta') + \sum_{n=1}^{\infty} q_n \{P_n(\mu) q_n(\zeta) + (-)^n P_n(\mu') q_n(\zeta')\} \quad . \quad . \quad (17)$$

where a_n is a function of c, by reason of symmetry. The potential must be constant over both spheroids, and already has the requisite behaviour at infinity. The factor $(-)^n$ compensates for the fact that the similarly charged contiguous surfaces of the spheroids are not in corresponding positions with respect to the origin. A harmonic series given by one spheroid corresponds to a series for the other with $P_n(\mu)$ replaced, not by $P_n(\mu')$, but by $P_n(-\mu')$ or $(-)^n P_n(\mu')$. The expression above is appropriate for the region outside, on the positive side of the z-axis. By the obvious symmetry, it is sufficient to make the above form of V constant on the spheroid of centre O_2, the origin.

By the transformation formula for $P_n(\mu') q_n(\zeta')$, we have

$$V = q_0(\zeta) + \frac{\pi}{2}\sum_0^{\infty}(-)^r(2r+1)P_r(\mu)p_r(\zeta)K_r^0\left(\frac{c}{a}\right) + \sum_{r=1}^{\infty} a_r P_r(\mu) q_r(\zeta)$$
$$+ \frac{\pi}{2}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}(-)^{r+n}a_n(2r+1)P_r(\mu)p_r(\zeta)K_r^n\left(\frac{c}{a}\right)$$

and this is constant when $\zeta = \zeta_0$ provided that, for all values of r, except zero,

$$\frac{2}{\pi}a_r q_r(\zeta_0) + (-)^r(2r+1)p_r(\zeta_0)K_r^0\left(\frac{c}{a}\right) + (-)^r(2r+1)p_r(\zeta_0)\sum_{n=1}^{\infty}a_n(-)^n K_r^n\left(\frac{c}{a}\right) = 0,$$

or, introducing a coefficient a_0, equal to unity,

$$a_0 = 1$$
$$-a_r \frac{q_r(\zeta_0)}{p_r(\zeta_0)}\frac{(-)^r}{2r+1} = \frac{\pi}{2}\sum_{n=0}^{\infty}(-)^n a_n K_r^n\left(\frac{c}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and the problem is reduced to a determination of the coefficients a_r satisfying the infinity of equations thus involved. But K_r^n is symmetrical in r and n, and, moreover, $K_r^n(c/a)$ is of order $(a/c)^{n+r+1}$ when c/a is large, in which case the values of a_r rapidly decrease as r increases, and can be evaluated readily by successive approximations. For our purposes, however, they are not very interesting in the case of spheroids, and we shall confine further developments to the case of two equal parallel charged circular discs, corresponding to $\zeta_0 = 0$.

§ 7. *Case of Two Charged Circular Discs.*

When the spheroids become discs, and $\zeta_0 = 0$, we have $p_r(\zeta_0) = 0$ if r is odd. Thus

$$-a_r = (-)^r(2r+1)\frac{\pi}{2}\frac{p_r(\zeta_0)}{q_r(\zeta_0)}\sum_0^{\infty}a_n(-)^n K_r^n\left(\frac{c}{a}\right) = 0$$

if r is odd. Moreover,

$$p_{2m}(0)/q_{2m}(0) = 2/\pi$$

and we are only concerned with the coefficients a_{2m} satisfying

$$a_{2m} = -(4m+1)\sum_{n=0}^{\infty} a_{2n}\mathrm{K}_{2m}{}^{2n}\left(\frac{c}{a}\right), \qquad a_0 = 1. \quad . \quad . \quad . \quad . \quad (19)\ (\mathrm{A})$$

The potentials of the two discs are

$$(\mathrm{V}_1, \mathrm{V}_2) = (\mathrm{V})_{\zeta=\zeta_0=0}$$

and at any external point

$$\mathrm{V} = q_0(\zeta) + q_0(\zeta') + \sum_1^{\infty} a_{2n}\{q_{2n}(\zeta)\,\mathrm{P}_{2n}(\mu) + q_{2n}(\zeta')\,\mathrm{P}_{2n}(\mu')\}. \quad . \quad . \quad (20)$$

We may describe (A) as the fundamental equation of the problem. By analysis of some elegance, we shall replace it by an integral equation. This analysis brings out, in a striking way, the essential relation between the Bessel-Fourier type and the harmonic type of analysis of such problems, which has not hitherto been perceived in the absence of the formula (B) below, which is of a very essential character.

§ 8. *Derivation of an Integral Equation.*

We proved in a preceding section that for the origin O_1 $(z = -c)$

$$\mathrm{P}_n(\mu')\,q_n(\zeta') = (-)^n\sqrt{\frac{\pi}{2}}\cdot\frac{\mathrm{J}_{n+\frac{1}{2}}(\mathrm{D})}{\sqrt{\mathrm{D}}}\cdot\frac{a}{\mathrm{O}_1\mathrm{P}}$$

where $\mathrm{D} = a\partial/\partial c$, and in cylindrical co-ordinates, P is any point (z, ρ),

$$\mathrm{O}_1\mathrm{P}^2 = (z+c)^2 + \rho^2.$$

Now

$$\frac{1}{\mathrm{O}_1\mathrm{P}} = \int_0^{\infty} e^{-\lambda(z+c)}\,\mathrm{J}_0(\lambda\rho)\,d\lambda,$$

and therefore

$$\mathrm{P}_n(\mu')\,q_n(\zeta') = \left(\frac{\pi a}{2}\right)^{\frac{1}{2}}\frac{\mathrm{J}_{n+\frac{1}{2}}(a\partial/\partial c)}{(\partial/\partial c)}\int_0^{\infty} e^{-\lambda(z+c)}\,\mathrm{J}_0(\lambda\rho)\,d\lambda,$$

or

$$\mathrm{P}_n(\mu')\,q_n(\zeta') = \left(\frac{\pi a}{2}\right)^{\frac{1}{2}}\int_0^{\infty} e^{-\lambda(z+c)}\,\mathrm{J}_0(\lambda\rho)\,\mathrm{J}_{n+\frac{1}{2}}(\lambda a)\,\frac{\partial\lambda}{\sqrt{\lambda}}. \quad . \quad . \quad . \quad . \quad (21)\ (\mathrm{B})$$

In particular, with $c = 0$,

$$\mathrm{P}_n(\mu)\,q_n(\zeta) = \left(\frac{\pi a}{2}\right)^{\frac{1}{2}}\int_0^{\infty} e^{-\lambda z}\,\mathrm{J}_0(\lambda\rho)\,\mathrm{J}_{n+\frac{1}{2}}(\lambda a)\,\frac{d\lambda}{\sqrt{\lambda}}.$$

The more special cases $\mu = 0$, $\zeta = 0$ give two of SCHAFHEITLIN's formulæ.

Accordingly, the potential of the region outside the discs is

$$V = \sum_{0}^{\infty} a_{2m} \{ q_{2m}(\zeta) P_{2m}(\mu) + q_{2m}(\zeta') P_{2m}(\mu') \}$$

$$= \left(\frac{\pi a}{2}\right)^{\frac{1}{2}} \int_0^\infty e^{-\lambda z}(1 + e^{-\lambda c}) J_0(\lambda\rho) \frac{d\lambda}{\sqrt{\lambda}} \sum_{0}^{8} a_{2m} J_{2m+\frac{1}{2}}(\lambda a) \quad . \quad . \quad . \quad . \quad (22)$$

where, except when $m = 0$, in which case $a_0 = 1$,

$$a_{2m} = -(4m+1) \sum_{n=0}^{\infty} a_{2n} K_{2m}{}^{2n}\left(\frac{c}{a}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

or

$$a_{2m} = -(4m+1) \sum_{n=0}^{\infty} a_{2n} \int_0^\infty e^{-cx/a} J_{2m+\frac{1}{2}}(x) J_{2n+\frac{1}{2}}(x) \frac{dx}{x}$$

(recalling the value of $K_{2m}{}^{2n}$ as a definite integral). Thus,

$$a_{2m} = -(4m+1) \int_0^\infty e^{-cx/a} J_{2m+\frac{1}{2}}(x) \frac{dx}{x} \left\{ \sum_{n=0}^{\infty} a_{2n} J_{2n+\frac{1}{2}}(x) \right\}.$$

Now write, for brevity,

$$f(x) \equiv \sum_{0}^{\infty} a_{2n} J_{2n+\frac{1}{2}}(x),$$

so that, if $f(x)$ is found, the ultimate value of V is

$$V = \left(\frac{\pi a}{2}\right)^{\frac{1}{2}} \int_0^\infty e^{-\lambda z}(1 + e^{-\lambda c}) J_0(\lambda\rho) f(a\lambda) \frac{d\lambda}{\sqrt{\lambda}}, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and also

$$a_{2m} = -(4m+1) \int_0^\infty e^{-cx/a} J_{2m+\frac{1}{2}}(x) f(x) \frac{dx}{x}$$

where m can take all integer values 1, 2, 3, . . . to infinity, and $a_0 = 1$.

Now let y be any new variable independent of x, and multiply both sides of the equation by $J_{2m+\frac{1}{2}}(y)$, afterwards summing for all values of m.

We thus find

$$\sum_{1}^{\infty} a_{2m} J_{m+\frac{1}{2}}(y) = -\int_0^\infty e^{-cx/a} f(x) \frac{dx}{x} . S(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where

$$S(x) = \sum_{0}^{\infty} (4m+1) J_{2m+\frac{1}{2}}(x) J_{2m+\frac{1}{2}}(y).$$

The left-hand side is equal to

$$f(y) - a_0 J_{\frac{1}{2}}(y) = f(y) - J_{\frac{1}{2}}(y),$$

and, finally, $f(x)$ satisfies the equation

$$f(y) - J_{\frac{1}{2}}(y) = -\int_0^\infty e^{-cx/a} f(x) \frac{dx}{x} \sum_{0}^{\infty} (4m+1) J_{2m+\frac{1}{2}}(x) J_{2m+\frac{1}{2}}(y). \quad . \quad . \quad (26)$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

This is a homogeneous integral equation for the determination of $f(x)$. In order to perform the summation involved in the integrand, we have recourse to a theorem of GEGENBAUER,* of which a special case is the expansion

$$J_{\frac{1}{2}}(x+y)/(x+y)^{\frac{1}{2}} = (\pi/2xy)^{\frac{1}{2}} \sum_0^\infty (-)^s (2s+1) J_{s+\frac{1}{2}}(x) J_{s+\frac{1}{2}}(y),$$

or

$$\sum_0^\infty (-)^s (2s+1) J_{s+\frac{1}{2}}(x) J_{s+\frac{1}{2}}(y) = \frac{2\sqrt{(xy)}}{\pi} \cdot \frac{\sin(x+y)}{x+y} \quad . \quad . \quad . \quad . \quad (26\text{A})$$

Either of the variables may be negative, so that if $x \neq y$,

$$\sum_0^\infty (2s+1) J_{s+\frac{1}{2}}(x) J_{s+\frac{1}{2}}(y) = \frac{2\sqrt{(xy)}}{\pi} \cdot \frac{\sin(x-y)}{x-y}, \quad . \quad . \quad . \quad . \quad (27)$$

whence, by addition,

$$S(x) = \sum_0^\infty (4m+1) J_{2m+\frac{1}{2}}(x) J_{2m+\frac{1}{2}}(y)$$
$$= \frac{\sqrt{(xy)}}{\pi} \left\{ \frac{\sin(x+y)}{x+y} + \frac{\sin(x-y)}{x-y} \right\}. \quad . \quad . \quad . \quad . \quad (28)$$

Thus our integral equation takes the form

$$f(y) - J_{\frac{1}{2}}(y) = -\frac{\sqrt{y}}{\pi} \int_0^\infty e^{-cx/a} f(x) \frac{dx}{\sqrt{x}} \left\{ \frac{\sin(x+y)}{x+y} + \frac{\sin(x-y)}{x-y} \right\}. \quad . \quad . \quad (29)$$

Certain of the processes involved in this reduction would need, for completeness, a somewhat long discussion of convergence, which would take us far from the main theme of this memoir. Such details may, however, be left to the reader, it being sufficient to say that such justification is possible.

The reduction of a problem to the solution of an integral equation is always a matter of much interest, for the integral equation involves all the mathematical properties of the function needed for the exact solution, and reduces the various physical definitions of the function to one single equation. It can, in this sense, for many purposes, play the part of an exact solution.

Certain integral equations with the kernel

$$\sin(x+y)/(x+y)$$

have been discussed by HARDY, but the exponential factor entirely alters the type, and his results cannot be applied to this problem.

We notice that when $c \to \infty$, so that only the disc of centre O_2 remains, the expressions on the two sides vanish together if

$$f(y) = J_{\frac{1}{2}}(y).$$

* Math. Ann., II, 1871; WATSON, 'Theory of Bessel Functions,' p. 525.

Then

$$f(a\lambda) = \left(\frac{2}{\pi a\lambda}\right)^{\frac{1}{2}} \sin a\lambda$$

and the potential is

$$V = \int_0^\infty e^{-\lambda z} J_0(\lambda\rho) \frac{\sin \lambda a \, d\lambda}{\lambda},$$

which is the appropriate form for a freely charged disc.

§ 9. *Approximate Solutions.*

A short account of the use of the equation in developing a solution by successive approximation is desirable at this point, as certain general properties of the exact solution appear at the same time.

The value of the integral

$$I = \int_0^\infty e^{-kx} \left\{\frac{\sin(x+y)}{x+y} + \frac{\sin(x-y)}{x-y}\right\} dx$$

is readily found. Writing

$$I = \int_0^\infty e^{-kx} \, dx \int_0^1 d\alpha \, \{\cos \alpha(x+y) + \cos \alpha(x-y)\}$$

we may invert the order, if k is not zero, and find

$$I = \int_0^1 2 \cos \alpha y \, d\alpha \int_0^\infty e^{-kx} \cos \alpha x \, dx$$

$$= 2k \int_0^1 \frac{\cos \alpha y}{k^2 + \alpha^2} \, d\alpha.$$

The integral is of order k^{-1} in k (or c/a), and it is possible to differentiate continually with respect to k. If $D \equiv \partial/\partial k$,

$$D^n I = (-)^n \int_0^\infty x^n e^{-kx} \left\{\frac{\sin(x+y)}{x+y} + \frac{\sin(x-y)}{x-y}\right\} dx$$

$$= 2(-)^n D^n k \int_0^1 \frac{\cos \alpha y \, d\alpha}{k^2 + \alpha^2}.$$

We shall now suppose that $f(x)/\sqrt{x}$ admits an absolutely convergent expansion of the form

$$\frac{f(x)}{\sqrt{x}} = \sum_0^\infty b_n x^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

—a supposition justified by inspection of the form finally obtained—for finite values of x, and find

$$\int_0^\infty e^{-kx} f(x) \frac{dx}{\sqrt{x}} \left\{ \frac{\sin(x+y)}{x+y} + \frac{\sin(x-y)}{x-y} \right\} = \sum_0^\infty (-)^n b_n \mathrm{D}^n \mathrm{I}$$

$$= 2 \sum_0^\infty (-)^n b_n \mathrm{D}^n k \int_0^1 \frac{\cos \alpha y \, d\alpha}{k^2 + \alpha^2}.$$

The integral equation then becomes

$$\frac{f(y)}{\sqrt{y}} - \sqrt{\frac{2}{\pi}}, \qquad \frac{\sin y}{y} = -\frac{2}{\pi} \sum_0^\infty (-)^n b_n \mathrm{D}^n k \int_0^1 \frac{\cos \alpha y \, d\alpha}{k^2 + \alpha^2},$$

or, for all values of y,

$$\sum_0^\infty b_n y^n - \sqrt{\frac{2}{\pi}} \sum_0^\infty \frac{(-)^n y^{2n}}{2n+1\,!} = \frac{-2}{\pi} \sum_0^\infty (-)^n b_n \mathrm{D}^n k \int_0^1 \frac{d\alpha}{k^2+\alpha^2} \left(1 - \frac{\alpha^2 y^2}{2} \ldots \right).$$

It is at once evident that only even powers of y can occur in the series

$$f(y) = \sqrt{y} \sum_0^\infty b_n y^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and therefore $f(x)/\sqrt{x}$ is an even function of x.

This was also clear from its original definition, but requires some emphasis later. For such even powers, we have, including the case $m = 0$,

$$b_{2m} = (-)^n \sqrt{\frac{2}{\pi}} \cdot \frac{1}{(2m+1)\,!} = -\frac{(-)^m}{2m\,!} \frac{2}{\pi} \sum_0^\infty b_{2n} \mathrm{D}^{2n} k \int_0^1 \frac{\alpha^{2m} \, d\alpha}{k^2 + \alpha^2},$$

where

$$f(x)/\sqrt{x} = \sum_0^\infty b_{2m} x^{2m}.$$

The coefficients b_{2m} can be determined rapidly, by this formula, to any desired order of a/c or x^{-1}. For if $k > 1$,

$$\mathrm{D}^n k \int_0^1 \frac{\alpha^{2m}}{\alpha^2 + k^2} d\alpha = \frac{2n\,!}{2m+1} \frac{1}{k^{2n+1}} - \frac{(2n+2)\,!}{2\,!\,(2m+3)} \frac{1}{k^{2n+3}} + \ldots .$$

Thus,

$$b_{2m} - \frac{(-)^m}{(2m+1)\,!} \sqrt{\frac{2}{\pi}} = \frac{(-)^m}{2m\,!} \frac{2}{\pi} \sum_0^\infty b_{2n} \left\{ \frac{2n\,!}{2m+1} \frac{1}{k^{2n+1}} - \ldots \right\},$$

and we see that, to the third order in k^{-1} or a/c, for example,

$$b_0 - \sqrt{\frac{2}{\pi}} = \frac{2b_0}{\pi k}\left(1 - \frac{1}{3k^2}\right) + \frac{4b_2}{\pi k^3}$$

$$b_2 + \tfrac{1}{6} \sqrt{\frac{2}{\pi}} = -\frac{b_0}{\pi k}\left(\tfrac{1}{3} - \frac{1}{5k^2}\right) - \frac{2b_2}{3\pi k^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

In general, even to the order k^{-2} inclusive, we may prove

$$b_{2m} = \frac{(-)^m}{2m+1\,!}\sqrt{\frac{2}{\pi}}\left(1 - \frac{2}{\pi k}\right)^{-1},$$

which shows the rapid convergence of the series. Thus, for discs at all far apart, a very good value of $f(a\lambda)$ and thence of V is readily obtained for practical purposes, but we do not wish to develop it further from this point of view.

§ 10. *Potential in Series of Spherical Harmonics.*

If $f(a\lambda)$ has been found in the form

$$f(a\lambda) = \sqrt{(a\lambda)}\sum_0^\infty a_{2m}(a\lambda)^{2m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

we can readily pass to a development of V in spherical, instead of spheroidal, harmonics. For

$$\begin{aligned} V &= \sqrt{\left(\frac{\pi a}{2}\right)}\int_0^\infty e^{-\lambda z}(1 + e^{-\lambda c})\,J_0(\lambda\rho)\,f(a\lambda)\frac{d\lambda}{\sqrt{\lambda}} \\ &= a\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\lambda z}(1 + e^{-\lambda c})\,J_0(\lambda\rho)\,d\lambda\sum_0^\infty b_{2m}(a\lambda)^{2m} \\ &= a\sqrt{\frac{\pi}{2}}\sum_0^\infty b_{2m}(aD)^{2m}\int_0^\infty \{e^{-\lambda z} + e^{-\lambda(z+c)}\}\,J_0(\lambda\rho)\,d\lambda, \end{aligned}$$

where $D \equiv \partial/\partial z$, and at any point P, if r, R are the distances O_2P, O_1P,

$$V = a\sqrt{\frac{\pi}{2}}\sum_0^\infty b_{2m}(aD)^{2m}\left\{\frac{1}{r} + \frac{1}{R}\right\}.$$

But

$$\frac{\partial^n}{\partial z^n}\cdot\frac{1}{r} = (-)^n\frac{n\,!}{r^{n+1}}P_n(\mu),$$

and therefore

$$V = a\sqrt{\frac{\pi}{2}}\sum_0^\infty 2m\,!\,b_{2m}a^{2m}\left\{\frac{P_{2m}(\cos\theta)}{r^{2m+1}} + \frac{P_{2m}(\cos\theta')}{R^{2m+1}}\right\} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where (θ, θ') are the inclinations of O_2P, O_1P to the z-axis.

This form of solution, thus deducible at once from the integral equation, is especially useful in giving the distant field due to the two discs.

The total charge on each disc is evidently $Q = ab_0\sqrt{(\pi/2)}$, and the exact value of b_0 is given later. The approximate value is

$$\begin{aligned} Q = a\sqrt{\frac{\pi}{2}}\,.\,b_0 &= a\sqrt{\frac{\pi}{2}}\,.\,\sqrt{\frac{2}{\pi}}\left(1 - \frac{2}{\pi k}\right)^{-1} \\ &= a\left(1 - \frac{2a}{\pi c}\right)^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35) \end{aligned}$$

to order $(a/c)^2$ inclusive, where a is the radius of a disc, and c the distance apart.

Part II.—Preliminary Study of a Type of Integral Equation.

§ 11. *A Property of a Certain Symmetrical Function.*

The kernel of the integral equation, which we shall denote by

$$K(x, y) = \frac{\sin(x+y)}{x+y} + \frac{\sin(x+y)}{x-y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

possesses the remarkable property

$$\int_0^\infty K(x, y)\, K(y, t)\, dy = \pi K(x, t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

For consider the integral

$$I_1 = \int_0^\infty \frac{\sin(x+y)}{x+y} \cdot \frac{\sin(y+t)}{y+t}\, dy.$$

Then by obvious steps,

$$I_1 = \frac{1}{t-x}\int_0^\infty \left(\frac{1}{y+x} - \frac{1}{y+t}\right)\sin(y+x)\sin(y+t)\, dy$$

$$= \frac{1}{t-x}\int_x^\infty \frac{d\lambda}{\lambda}\sin\lambda\sin(\lambda+t-x) - \frac{1}{t-x}\int_t^\infty \frac{d\lambda}{\lambda}\sin\lambda\sin(\lambda-t+x),$$

changing the variable differently in the two portions. Again, changing the signs of x and t,

$$I_2 = \int_0^\infty \frac{\sin(y-x)}{y-x} \cdot \frac{\sin(y-t)}{y-t}\, dy$$

$$= \frac{1}{x-t}\int_{-x}^\infty \frac{d\lambda}{\lambda}\sin(\lambda-t+x) - \frac{1}{x-t}\int_{-t}^\infty \frac{d\lambda}{\lambda}\sin\lambda\sin(\lambda+t-x),$$

and by a simple reduction, we find

$$I_1 + I_2 = \frac{2}{t-x}\int_0^\infty \frac{d\lambda}{\lambda}\sin\lambda\,\{\sin\lambda+t-x) - \sin(\lambda-t+x)\}$$

$$= \frac{4}{t-x}\int_0^\infty \frac{d\lambda}{\lambda}\sin(t-x)\sin\lambda\cos\lambda$$

$$= \frac{2}{t-x}\sin(t-x)\int_0^\infty \frac{d\lambda}{\lambda}\sin 2\lambda = \pi\,\frac{\sin(t-x)}{t-x}.$$

The value of

$$I_3 + I_4 = \int_0^\infty \left\{\frac{\sin(x+y)}{x+y} \cdot \frac{\sin(y-t)}{y-t} + \frac{\sin(y+t)}{y+t} \cdot \frac{\sin(x-y)}{x-y}\right\} dy$$

may be deduced by changing the sign either of x or of t.

Thus the whole value of the integral

$$\int_0^\infty K(x, y) K(y, t)\, dt$$

is

$$I_1 + I_2 + I_3 + I_4 = \pi \left\{\frac{\sin(x+t)}{x+t} + \frac{\sin(x-t)}{x-t}\right\} = \pi K(x, t)$$

as stated.

§ 12. *Homogeneous Equations with* K (x, y) *as Kernel.*

Consider a homogeneous integral equation

$$\phi(y) = \lambda \int_0^\infty K(x, y)\, \phi(x)\, dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

with K (x, y) as its kernel, and λ being constant. Without recourse to very general theorems,* we can at once determine under what circumstances it has solutions. For if ϕ is any solution, we have, multiplying each side by K (y, t) and integrating from $y = 0$ to $y = \infty$,

$$\int_0^\infty \phi(y) K(y, t)\, dy = \lambda \int_0^\infty \phi(x)\, dx \int_0^\infty K(x, y) K(y, t)\, dy,$$

—assuming the inversion of order, which limits the types of solution concerned, but not in a manner relevant to our purpose.

Thus

$$\frac{1}{\lambda}\phi(t) = \lambda\pi \int_0^\infty \phi(x) K(x, t)\, dt = \pi\phi(t)$$

or $\lambda = 1/\pi$, the only alternative being $\phi(t) = 0$. Thus the only non-zero solutions occur when $\lambda = 1/\pi$. For our study, the significance of the result lies in the fact that the homogeneous equation has no solution when $\lambda = -1/\pi$, which is the value in the electrical problem.

When $\lambda = 1/\pi$ an infinite number of solutions exist, of very diverse types, some of which may be indicated. For example, a formula of SONINE is

$$J_0(k) = \frac{2}{\pi}\int_0^\infty J_0(x) \frac{\sin(x+k)}{x+k}\, dx$$

which is equivalent to

$$J_0(k) = \frac{1}{\pi}\int_0^\infty J_0(x) K(x, k)\, dx$$

so that $J_0(y)$ is a solution of the homogeneous equation with $\lambda = 1/\pi$.

* *Vide, e.g.*, WHITTAKER and WATSON, 'Modern Analysis,' Camb. Univ. Press.

There are important "orthogonal functions" connected with the equation. For example, it is satisfied by

$$\phi(y) = J_{\frac{1}{2}}(y)/\sqrt{y}$$

for

$$\int_0^\infty K(x, y)\, J_{\frac{1}{2}}(x) \frac{dx}{\sqrt{x}} = \tfrac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty K(x, y)\, K(x, 0)\, dx$$

$$= \sqrt{\frac{\pi}{2}} \,.\, K(y, 0) = \pi J_{\frac{1}{2}}(y)/\sqrt{y}$$

and we may enquire under what circumstances the function

$$\phi(y) = J_{n+\frac{1}{2}}(y)/\sqrt{y}$$

is a solution. Consider the integral

$$I = \int_9^\infty K(x, y)\, J_{n+\frac{1}{2}}(x) \frac{dx}{\sqrt{x}}.$$

Then expanding the kernel,

$$I = \int_0^\infty \frac{2\pi}{\sqrt{y}} \,.\, J_{n+\frac{1}{2}}(x) \frac{dx}{x} \sum_0^\infty (4r+1)\, J_{2r+\frac{1}{2}}(x)\, J_{2r+\frac{1}{2}}(y)$$

$$= \frac{2\pi}{\sqrt{y}} \sum_0^\infty (4r+1)\, J_{2r+\frac{1}{2}}(x)\, I_r$$

where

$$I_r = \int_0^\infty J_{n+\frac{1}{2}}(x)\, J_{2r+\frac{1}{2}}(x) \frac{dx}{x}.$$

Now by a very general formula of SCHAFHEITLIN, when all numbers are real, and

$$\sigma < 0, \qquad \nu + \rho + \sigma > -1,$$

$$\int_0^\infty J_\nu(x)\, J_\rho(x)\, x^\sigma dx = \frac{2^\sigma\, \Gamma(-\sigma)\, \Gamma\{\frac{1}{2}(1+\nu+\rho+\sigma)\}}{\Gamma\{\frac{1}{2}(1+\nu-\rho-\sigma)\}\, \Gamma\{\frac{1}{2}(1+\rho-\nu-\sigma)\}\, \Gamma\{\frac{1}{2}(1+\nu+\rho-\sigma)\}}.$$

In our special case $\sigma = -1$, and on reduction

$$I_r = \frac{1}{\pi} \cdot \frac{\sin \frac{1}{2}\pi(n-2r)}{(n-2r)(n+2r+1)}.$$

Thus

$$I = \frac{2}{\sqrt{y}} \sum_0^\infty \frac{4r+1}{(n-2r)(n+2r+1)} J_{2r+\frac{1}{2}}(y) \sin(n-2r)\frac{\pi}{2}$$

If n is even, all the terms of this series vanish except that for which $r = 2n$, and its value is $\pi J_{2n+\frac{1}{2}}(y)/\sqrt{y}$.

Accordingly,

$$\int_0^\infty J_{2n+\frac{1}{2}}(x) \frac{dx}{\sqrt{x}} K(x, y) = \frac{\pi}{\sqrt{y}} J_{2n+\frac{1}{2}}(y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and if n is an integer, $J_{2n+\frac{1}{2}}(y)/\sqrt{y}$ is a solution of the integral equation. On the other hand, if $2n$ is replaced by an odd integer, it is not a solution.

Moreover, since by the above equations, if $n \neq r$,

$$\int_0^\infty J_{2n+\frac{1}{2}}(y)\, J_{2r+\frac{1}{2}}(y)\frac{dy}{y} = 0,$$

it follows that $\{J_{2n+\frac{1}{2}}(y),\, J_{2r+\frac{1}{2}}(y)\}/\sqrt{y}$ are orthogonal functions.

A consequence of the above analysis is that any function—which must be even—capable of expansion in a convergent Neumann series of the form

$$f(x) = \sum_0^\infty a_n J_{2n+\frac{1}{2}}(x)/\sqrt{x} \quad \text{. . . . .} \quad (40)$$

is a solution of the integral homogeneous equation with $\lambda = 1/\pi$.

Characteristic examples are $J_{2m}(x)$, $\cos \lambda x$, x^{2m}, where m is an integer.

§ 13. *The Integral Equation of the Electrical Problem.*

The integral equation

$$\frac{f(y)}{\sqrt{y}} - \frac{J_{\frac{1}{2}}(y)}{\sqrt{y}} = -\frac{1}{\pi}\int_0^\infty f(x)\, e^{-\eta x}\, dx \mathrm{K}(x, y) \quad \text{. . . . . . .} \quad (41)$$

where $\eta = c/a$, arising in the electrical problem, may be written

$$\phi(y) - \frac{J_{\frac{1}{2}}(y)}{\sqrt{y}} = -\frac{1}{\pi}\int_0^\infty \phi(x)\, e^{-\eta x}\, \mathrm{K}(x, y)\, dx \quad \text{. . . . . . .} \quad (42)$$

where $\phi(y) = f(y)/\sqrt{y}$. We could choose a new kernel $e^{-\eta(x+y)}\mathrm{K}(x, y)$, but this would make the form less advantageous, as an attempt to solve the equation in this manner has shown. We accordingly retain $\mathrm{K}(x, y)$ as kernel.

Now $\phi(y)$ is, by the definition originally of $f(y)$, expansible in the form

$$\Sigma a_n J_{2n+\frac{1}{2}}(y)/\sqrt{y},$$

as also is $J_{\frac{1}{2}}(y)/\sqrt{y}$. Thus by the preceding section, $\phi(y)$—$J_{\frac{1}{2}}(y)/\sqrt{y}$ is a solution of the homogeneous integral equation with $\lambda = +1/\pi$. Accordingly,

$$\phi(y) - J_{\frac{1}{2}}(y)/\sqrt{y} = \frac{1}{\pi}\int_0^\infty \{\phi(x) - J_{\frac{1}{2}}(x)/\sqrt{x}\}\, \mathrm{K}(x, y)\, dx.$$

By subtraction from (42) we find

$$\int_0^\infty \{\phi(x) - J_{\frac{1}{2}}(x)/\sqrt{x} + \phi(x)\, e^{-\eta x}\}\, \mathrm{K}(x, y) = 0 \quad \text{. . . . . .} \quad (43)$$

or

$$\phi(x)\{1 + e^{-\eta x}\} = J_{\frac{1}{2}}(x)/\sqrt{x} + \psi(x),$$

where $\psi(x)$ is any solution of

$$\int_0^\infty \psi(x)\,K(x, y)\,dx = 0. \qquad (44)$$

A knowledge of the appropriate $\psi(x)$ would give a complete solution of the problem.

§ 14. *The Equation.*

$$\int_0^\infty \psi(x)\,K(x, y)\,dx = 0.$$

We must now consider the types of solution which this equation yields, noticing, in the first place, that since $\phi(x)$ is an even function of x, as already shown, $\phi(x)\{1 + e^{-\eta x}\}$ is a mixed function (neither odd nor even). $J_{\frac{1}{2}}(x)/\sqrt{x}$ is even, and therefore $\psi(x)$ is mixed, or has both odd and even terms.

The equation has an infinite number of types of solution. None can be even functions of x expressible as a Neumann series, for all such functions, as we have seen, satisfy the homogeneous equation.

Whether $\psi(x)$ be odd or even, the integral is necessarily an even function of y, for it is unchanged when the sign of y is changed. In fact, for any $\psi(x)$ the integral

$$F(y) = \int_0^\infty \psi(x)\,K(x, y)\,dx$$

is necessarily even in y.

Any number of solutions can be found as follows, involving both odd and even functions of y:—

Let $F_1(x)$ be any odd function of x giving a finite integral

$$G_1(y) = \int_0^\infty F_1(x)\,K(x, y)\,dy,$$

where G_1 is even in y—and therefore over a wide range of forms of G_1 expressible as a Neumann series—and a solution of the homogeneous equation

$$G_1(y) = \frac{1}{\pi}\int_0^\infty G_1(x)\,K(x, y)\,dx.$$

Then by subtraction,

$$\int_0^\infty \left\{F_1(x) - \frac{1}{\pi}G_1(x)\right\} K(x, y)\,dx = 0,$$

and $F_1 - G_1/\pi$ is a solution of the present equation.

Thus, if F_1 covers a wide range of types of odd function, a mixed solution is given by

$$F_1(x) - \frac{1}{\pi}\int_0^\infty F_1(y)\,K(x, y)\,dy. \qquad (45)$$

One or two illustrative cases may be quoted, all of which can be verified otherwise. It is useful to notice that

$$\mathrm{K}(x, y) = 2\int_0^1 \cos \alpha x \cos \alpha y \, d\alpha,$$

so that, when inversion is justified, as in the examples,

$$\int_0^\infty f(x)\,\mathrm{K}(x, y)\,dx = 2\int_0^1 \cos \alpha y \, d\alpha \int_0^\infty f(x) \cos \alpha x \, dx.$$

Thus the relation

$$\mathrm{F}(y) = \int_0^\infty f(x)\,\mathrm{K}(x, y)\,dx$$

is equivalent to

$$\mathrm{F}(y) = 2\int_0^1 \cos \alpha y \, d\alpha \int_0^\infty f(x) \cos \alpha x \, dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

showing an obvious relation to FOURIER's integral theorem.

Any solution of the equation, with $\alpha \ngtr 1$

$$\int_0^\infty \psi(x) \cos \alpha x \, dx = 0,$$

is, *ipso facto*, a solution of

$$\int_0^\infty \psi(x)\,\mathrm{K}(x, y)\,dx = 0.$$

For example, if $a > \alpha$, and Y_0 is the second Bessel function

$$\int_0^\infty \mathrm{Y}_0(ax) \cos \alpha x \, dx = 0,$$

and therefore, when a is greater than unity

$$\int_0^\infty \mathrm{Y}_0(ax)\,\mathrm{K}(x, y)\,dx = 0,$$

and more generally, if $a > 1$, and F is arbitrary,

$$\int_0^\infty dx\,\mathrm{K}(x, y) \int_a^\infty \mathrm{Y}_0(tx)\,\mathrm{F}(t)\,dt = 0,$$

giving a variety of logarithmic types of solution.

Proceeding to simpler illustrations, take the odd function $\mathrm{J}_1(ax)$. We know that

$$\int_0^\infty \mathrm{J}_1(ax) \cos \alpha x \, dx = \frac{1}{a} \qquad (a > \alpha)$$

$$= \frac{1}{a}\left(1 - \frac{\alpha}{\sqrt{(\alpha^2 - a^2)}}\right) \qquad (a < \alpha).$$

Therefore

$$\int_0^\infty \mathrm{J}_1(ax)\,\mathrm{K}(x, y)\,dx = 2\int_0^1 \cos \alpha y \, d\alpha \int_0^\infty \mathrm{J}_1(ax) \cos \alpha x \, dx.$$

If $a > 1$, then $a > \alpha$ over the whole range of α, and the last double integral becomes

$$2\int_0^1 \cos \alpha y\, d\alpha \,.\, \frac{1}{a} = \frac{2 \sin y}{ay}.$$

But if $\alpha > 1$, it becomes

$$\frac{2}{a}\int_0^a \cos \alpha y\, d\alpha + \frac{2}{a}\int_a^1 \cos \alpha y \left(\frac{\alpha}{\sqrt{(\alpha^2 - a^2)}}\right) d\alpha = \frac{2 \sin y}{ay} - \frac{2}{a}\int_a^1 \frac{\alpha \cos \alpha y\, d\alpha}{\sqrt{(\alpha^2 - a^2)}},$$

the two results being continuous when $a = 1$. In particular, if a and b both exceed unity,

$$\int_0^\infty \{a J_1(ax) - b J_1(bx)\}\, K(x, y)\, dx = 0,$$

and $aJ_1(ax) - bJ_1(bx)$ is a solution of our equation. This example is included because it shows that purely odd solutions are possible. The result remains true if a or b becomes equal to unity. Proceeding to the limit $a = b$, where $b \not< 1$, we find that $xJ_0(bx)$ is a solution, and from this we can derive an infinite number of purely odd solutions. For any function capable of an expansion

$$f(x) = \sum_0^\infty a_n J_0(nx)$$

must satisfy

$$\int_0^\infty x f(x)\, K(x, y)\, dx = 0.$$

More generally, with certain limitations, if $F(\lambda)$ is arbitrary, and $b \not< 1$, the integral

$$x\int_b^\infty J_0(\lambda x)\, F(\lambda)\, d\lambda$$

is a solution. Evidently the number of purely odd solutions is infinite.

The purely even solutions, which also exist, are important in that they are only possible for functions which do not possess a convergent development in a Neumann series. Any such function possessing this development must, as we have seen, satisfy the homogeneous equation

$$\phi(y) = \frac{1}{\pi}\int_0^\infty \phi(x)\, K(x, y)\, dx,$$

but we have no general result for this integral if $\phi(x)$ has not the property, and the integral may be zero. One instance will suffice. Consider the function

$$F(y) = \int_0^\infty \frac{x \sin x - q \cos x}{x^2 + q^2}\, K(x, y)\, dx$$

when q is not zero. It is equivalent to

$$F(y) = 2\int_0^1 \cos \alpha q\, d\alpha \int_0^\infty \frac{x \sin x - q \cos x}{x^2 - q^2}\,,\, \cos \alpha x\, dx$$

$$= \int_0^1 \cos \alpha y\, d\alpha \int_0^\infty dx \left\{\frac{x \sin(\alpha + 1)x + x \sin(1 + \alpha)x}{x^2 + q^2} - \frac{q \cos(\alpha + 1)x + q \cos(1 - \alpha)x}{x^2 + q^2}\right\}$$

which, quoting the well-known forms

$$\int_0^\infty \frac{\cos rx}{q^2+x^2}\,dx = \frac{\pi}{2q}\,e^{-rq}, \qquad \int_0^\infty \frac{x \sin rx}{q^2+x^2}\,dx = \frac{\pi}{2}\,e^{-rq}$$

is at once seen to be zero. Thus, a purely even solution is

$$(x \sin x - q \cos x)/(x^2+q^2)$$

when q is not zero. If q is zero, it suddenly satisfies the homogeneous equation instead.

Mixed solutions occur in cases for which the individual odd and even parts are not solutions. One is evident already in the form, from analysis of this section,

$$\int_0^\infty \left\{\frac{\sin x}{x} - \tfrac{1}{2}a\pi \mathrm{J}_1(ax)\right\} \mathrm{K}(x, y)\,dx = 0$$

when a is not less than unity. The first portion here satisfies the homogeneous equation.

The general formula for a mixed solution has already been mentioned in this section, and is

$$\mathrm{F}(y) - \frac{1}{\pi}\int_0^\infty \mathrm{F}(x)\,\mathrm{K}(x, y)\,dx$$

where $\mathrm{F}(y)$ is an arbitrarily chosen odd function, such that the integral is finite—and also necessarily even. We can clearly modify this by the removal of the restriction that $\mathrm{F}(x)$ itself should be odd.

The most important example, to which attention will now be restricted with a view to future use, is

$$\mathrm{F}(x) = e^{-\eta x},$$

where η is any constant. With this value,

$$\int_0^\infty \mathrm{F}(x)\,\mathrm{K}(x, y)\,dx = 2\int_0^1 d\alpha \cos \alpha y \int_0^\infty e^{-\eta x} \cos \alpha x\,dx$$

$$= 2\int_0^1 \frac{\eta\,d\alpha \cos \alpha y}{\eta^2+\alpha^2},$$

and, therefore, for every positive value of η,

$$\int_0^\infty \left\{e^{-\eta x} - \frac{2}{\pi}\int_0^1 \frac{\eta\,d\alpha \cos \alpha x}{\eta^2+\alpha^2}\right\} \mathrm{K}(x, y)\,dx = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

The bracket vanishes when $\eta = o$ and when $\eta = \infty$. We may multiply by any function $\psi(\eta)$ of η which makes the bracket integrable from $\eta = 0$ to $\eta = \infty$, with a reversal of the order. For example, if $\psi(\eta) = \sin q\eta$, where q is not zero,

$$0 = \int_0^\infty \mathrm{K}(x, y)\, dx \int_0^\infty d\eta \left\{e^{-\eta x} \sin q\eta - \frac{2}{\pi} \int_0^1 \frac{\eta \sin q\eta \cos \alpha x\, d\alpha}{\eta^2 + \alpha^2}\right\}$$

$$= \int_0^\infty \mathrm{K}(x, y)\, dx \left\{\frac{q}{q^2 + x^2} - \frac{2}{\pi} \int_0^1 \cos \alpha x\, d\alpha\, \frac{\pi}{2} e^{-q\alpha}\right\}$$

$$= -e^{-q} \int_0^\infty \frac{x \sin x - q \cos x}{x^2 + q^2} \mathrm{K}(x, y)\, dx,$$

which verifies our previous purely even solution.

More generally, if, in seeking a mixed solution, we choose our initial function as $e^{-\eta x} f(x)$, where $f(x)$ is even—not necessarily, but for analytical convenience—we have

$$\int_0^\infty e^{-\eta x} f(x)\, \mathrm{K}(x, y)\, dx = f\left(\frac{\partial}{\partial \eta}\right) \int_0^\infty e^{-\eta x}\, \mathrm{K}(x, y)\, dx$$

$$= f\left(\frac{\partial}{\partial \eta}\right) \int_0^1 \frac{2\eta \cos \alpha y\, d\alpha}{\eta^2 + \alpha^2},$$

and the function

$$\psi(x) = f(x)\, e^{-\eta x} - \frac{1}{\pi} f \frac{\partial}{\partial \eta} 2\eta \int_0^1 \frac{\cos \alpha x\, d\alpha}{\eta^2 + \alpha^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

satisfies the equation

$$\int_0^\infty \psi(x)\, \mathrm{K}(x, y)\, dx = 0.$$

This is an important fundamental formula.

Now

$$\left(\frac{\partial}{\partial \eta}\right)^{2n} \frac{2\eta}{\eta^2 + \alpha^2} = \frac{2 \,.\, 2n\,!}{(\eta^2 + \alpha^2)^{n+\frac{1}{2}}} \cos (2n + 1) \tan^{-1} \frac{\alpha}{\eta},$$

and if

$$f(x) = \sum_0^\infty a_n x^{2n} e^{-\eta x},$$

we find, for all values of the coefficients a_n for which convergence is secured, that the function

$$\psi(x) = \sum_0^\infty a_n x^{2n} e^{-\eta x} - \frac{2}{\pi} \int_0^1 \cos \alpha x\, d\alpha \sum_0^\infty \frac{2n\,!\; a_n}{(\eta^2 + \alpha^2)^{n+\frac{1}{2}}} \cos (2n + 1) \tan^{-1} \frac{\alpha}{\eta} \quad . \quad (49)$$

is a solution of the equation

$$\int_0^\infty \psi(x)\, \mathrm{K}(x, y)\, dx = 0.$$

It is again to be emphasised that the "mixed" character of the function, together with the presence of the factor $e^{-\eta x}$, determines its importance.

Our digression into a general survey of a new type of integral equation has been made as brief as possible, with the limited scope of determining the features which bear on its application to our cardinal problem. In a strict mathematical sense, therefore, it is somewhat special, though the type of equation is worthy of more extended study. The factor e^{-kx} precludes any correspondence with HARDY'S discussion of integral equations of a more formal type with a kernel which is essentially $K(x, y)$.

§ 15. *The Cardinal Equation.*

We now proceed to indicate the method by which a solution of the original integral equation, suitable for our purpose, was ultimately found. In the original equation

$$\phi(y) - J_{\frac{1}{2}}(y)/\sqrt{y} = -\frac{1}{\pi}\int_0^\infty \phi(x)\, e^{-2qx}\, K(x, y)\, dx$$

where

$$\phi(x) = f(x)/\sqrt{x}, \qquad \eta = 2q = c/a,$$

the latter substitution being very convenient, we write

$$\phi(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{Q(x)}{2\cosh qx}$$

so that $Q(x)$, like $\phi(x)$, is an even function. Using the fact that $\phi(y) - J_{\frac{1}{2}}(y)/\sqrt{y}$ satisfies the homogeneous equation with coefficient π^{-1}, we obtain

$$\int_0^\infty \left\{\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{Q(x)}{2\cosh qx} - \frac{J_{\frac{1}{2}}(x)}{\sqrt{x}} + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{Q}{2\cosh qx} e^{-2qx}\right\} K(x, y)\, dx = 0$$

which reduces at once to

$$\int_0^\infty \left(Q(x)\, e^{-qx} - \frac{\sin x}{x}\right) K(x, y)\, dx = 0. \qquad (50)$$

This equation arises in any attempt to find a suitable solution, and we shall call it the *cardinal equation*. It is required to find an even solution $Q(x)$, not necessarily—nor in fact—expressible as a convergent series of power of x. The equation admits, indeed, an infinite number of purely odd, purely even, or mixed solutions, and must be combined with other conditions involved in the problem to obtain a unique suitable one—and especially combined with the original equation, which is more general.

§ 16. *Solution of the Cardinal Equation into an Expansion Theorem.*

The following solution is obtained by a symbolical method, which leaves something to be desired from the point of view of rigour. But there is no actual doubt regarding the final result. The direct form of the integral equation for $Q(x)$, which does not use the fact that $\phi(x)$ satisfies the homogeneous equation, is

$$\frac{Q(y)}{2\cosh qy} - \frac{\sin y}{y} = -\frac{2}{\pi}\int_0^\infty \frac{e^{-2qx}}{2\cosh qx} Q(x)\, K(x, y)\, dx.$$

Recalling that

$$\frac{\sin y}{y} = \frac{1}{\pi}\int_0^\infty \frac{\sin x}{x}\,\mathrm{K}(x, y)\,dx$$

we find

$$\frac{\mathrm{Q}(y)}{2\cosh qy} = -\frac{1}{\pi}\int_0^\infty \left(\mathrm{Q}(x)\,e^{-qx} - \frac{\sin x}{x}\right)\mathrm{K}(x, y)\,dx$$

and using the cardinal equation of the last section,

$$\frac{\mathrm{Q}(y)}{2\cosh qy} = \frac{1}{\pi}\int_0^\infty \mathrm{Q}(x)\,e^{-qx}\left\{1 - \frac{e^{-qx}}{2\cosh qx}\right\}\mathrm{K}(x, y)\,dx$$

or, briefly,

$$\int_0^\infty \frac{\mathrm{Q}(x)}{\cosh qx}\,\mathrm{K}(x, y)\,dx = \pi\,\frac{\mathrm{Q}(y)}{\cosh qy},$$

indicating that $\mathrm{Q}(x)/\cosh qx$ satisfies the homogeneous equation in its only soluble case. This is, in fact, only a verification, but it is useful as the condition to be superposed on the solution of the cardinal equation. We write accordingly, as before,

$$\sqrt{\frac{2}{\pi}}\,\mathrm{Q}(x) = 2\cosh qx \sum_0^\infty a_n \mathrm{J}_{2n+\frac{1}{2}}(x)/\sqrt{x} = 2\cosh qx\phi(x),$$

$$\phi = \sqrt{\frac{2}{\pi}}\,\mathrm{Q}(x)/2\cosh qx,$$

and the cardinal equation gives

$$2\sum_0^\infty a_n \int_0^\infty \cosh qx e^{-qx} \mathrm{J}_{2n+\frac{1}{2}}(x)\,\mathrm{K}(x, y)\,\frac{dx}{\sqrt{x}} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin x}{x}\,\mathrm{K}(x, y)\,dx.$$

Replacing $\mathrm{K}(x, y)$ by its equivalent,

$$2\sum_0^\infty a_n \int_0^\infty \cosh qx e^{-qx}\,\mathrm{J}_{2n+\frac{1}{2}}(x)\,\frac{dx}{\sqrt{x}}\int_0^1 \cos\alpha y\cos\alpha x\,d\alpha = \sqrt{\frac{2}{\pi}}\int_0^1 \cos\alpha y\,d\alpha\int_0^\infty \frac{\sin x}{x}\cos\alpha x\,dx,$$

or, removing the α-integration,

$$2\sum_0^\infty a_n \int_0^\infty \cosh qx e^{-qx}\mathrm{J}_{2n+\frac{1}{2}}(x)\,\frac{dx}{\sqrt{x}}\cos\alpha x = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin x}{x}\cos\alpha x\,dx = \sqrt{\frac{\pi}{2}} \quad . \quad . \quad . \quad (51)$$

where α can range from zero to unity. This is clearly, from the removal of the integration to α, not general, but if it leads to a solution of the required form, this can be the only physical solution. Briefly

$$2\int_0^\infty \cosh qx e^{-qx}\phi(x)\cos\alpha x\,dx = \sqrt{\frac{\pi}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

subject to somewhat stringent conditions. $\phi(x)$ must be an even function, and α must be between zero and unity, while $\phi(x)$ *must be independent of* α.

Even under these conditions, the equation has a variety of solutions. But the form of $\phi(x)$ we need for the problem of electrostatics possesses also the property of tending, when q tends to ∞, to $\sqrt{\frac{2}{\pi}} \cdot \frac{\sin x}{x}$, for this is the case in which $c/a \to \infty$, or only one electrified disc remains, for which the solution is known. Thus a necessary form of ϕ is

$$\phi(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin x}{x} \left\{1 + \frac{f_1(x)}{q} + \frac{f_2(x)}{q^2} + \dots\right\}, \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where all the f's are even functions independent of q, and also, of course, of α.

Now, if D is the operator $\partial/\partial\alpha$, we may write

$$2 \sum_0^\infty a_n \int_0^\infty \cosh qx e^{-qx} J_{2n+\frac{1}{2}}(x) \cos \alpha x \frac{dx}{\sqrt{x}} = \cos q \mathrm{D} \sum_0^\infty a_n \int_0^\infty e^{-qx} J_{2n+\frac{1}{2}}(x) \cos \alpha x \frac{dx}{\sqrt{x}},$$

and our equation becomes

$$2 \sum_0^\infty a_n \int_0^\infty e^{-qx} J_{2n+\frac{1}{2}}(x) \cos 2x \frac{dx}{\sqrt{x}} = \sqrt{\frac{2}{\pi}} \sec q \mathrm{D} \int_0^\infty \frac{\sin x \cos \alpha x}{x} dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin x \cos 2x}{x \cosh qx} dx, \quad (54)$$

the use of such operations being known to be legitimate for integrals of the type on the right. This integral can be evaluated, and the main difficulty at this point is the persistent factor e^{-qx} on the left, which is not an even function. If it were, a solution could be written down by inspection. In actual fact, a somewhat long investigation is necessary in order to get rid of this exponential, for, in the only mode of solution we have found, the value of the integral on the left is indicated explicitly.

One of our earlier formulæ was

$$\int_0^\infty e^{-qx} J_{2n+\frac{1}{2}}(x) \frac{dx}{\sqrt{x}} = \sqrt{\frac{2}{\pi}} q_{2n}(q)$$

in the present notation—the proof was valid for complex values of q with a positive real part. Accordingly,

$$2 \int_0^\infty e^{-qx} J_{2n+\frac{1}{2}}(x) \cos \alpha x \frac{dx}{\sqrt{x}} = \sqrt{\frac{2}{\pi}} \{q_{2n}(q + \iota\alpha) + q_{2n}(q - \iota\alpha)\} \quad . \quad . \quad (55)$$

—evidently in all circumstances an even function of α and an odd function of q, for $q_{2n}(q)$ is odd in q.

The equation to be solved is now

$$\sum_0^\infty a_{2n} \{q_{2n}(q + \iota\alpha) + q_{2n}(q - \iota\alpha)\} = \int_0^\infty \frac{\cos \alpha x}{\cosh qx} \frac{\sin x}{x} dx.$$

But we may write

$$\int_0^\infty \frac{\cos \alpha x}{\cosh qx} \frac{\sin x}{x} dx = \int_0^\infty \frac{\sin x}{x \cosh qx} \frac{dx}{2 \sinh qx} \{\sinh (q + \iota\alpha) x + \sinh (q - \iota\alpha) x\}$$

so that

$$\sum_0^\infty a_{2n} \{q_{2n}(q + \iota\alpha) + q_{2n}(q - \iota\alpha)\} = \int_0^\infty \frac{\sin x \, dx}{x \sinh 2qx} \{\sinh (q + \iota\alpha) x + \sinh (q - \iota\alpha) x\}. \quad (56)$$

This equation is satisfied identically if, for suitable values of a variable ρ, whose imaginary part does not exceed unity, and whose real part is less than $2q$, we can select coefficients a_{2n} in accordance with

$$\sum_0^\infty a_{2n} q_{2n}(\rho) = \int_0^\infty \frac{\sin \rho x}{\sinh 2qx} \frac{\sin x}{x} dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

where each side is an odd function of ρ. The use of this apparently simple artifice yields a definition of the coefficients which can in fact lead directly to a solution. The value of the integral can be deduced from that of an integral due to POISSON, namely,

$$\int_0^\infty \frac{\sinh px}{\sinh qx} \cos kx \, dx = \frac{\pi}{2q} \sin \frac{p\pi}{2q} \Big/ \left\{\cosh \frac{k\pi}{q} + \cos \frac{p\pi}{q}\right\}$$

where $|p| < |q|$. Writing ρ for p and $2q$ for q, and integrating with respect to k from zero to unity—this involves no delicate considerations if the preceding condition is satisfied—we find

$$\int_0^\infty \frac{\sin x}{x} \frac{\sinh \rho x}{\sinh 2qx} dx = \frac{\pi}{4q} \sin \frac{q\pi}{2q} \int_0^1 \frac{dk}{\cosh \frac{k\pi}{2q} + \cos \frac{\pi\rho}{2q}}.$$

When q tends to infinity, the right-hand side tends to

$$\frac{\pi}{4q} \cdot \frac{\rho\pi}{2q} \cdot \tfrac{1}{2}$$

and we have

$$\operatorname*{Lt}_{q \to \infty} \sum_0^\infty a_{2n} q_{2n}(\rho) = \pi^2 \rho / 16 q^2.$$

This is in violation of a fundamental property of our function ϕ where

$$\phi = \sum_0^\infty a_{2n} J_{2n+\frac{1}{2}}(x)/\sqrt{x},$$

for when $q \to \infty$, we have $a_0 \to 0$, $a_{2n} \to 0$ $(n \neq 0)$, so that the left-hand side should tend to $q_0(\rho)$ instead of zero. The solution we thus obtain is therefore not the solution sought, and we return to the equation

$$\sum_0^\infty a_{2n} \{q_{2n}(q + \iota\alpha) + q_{2n}(q - \iota\alpha)\} = \int_0^\infty \frac{\sin x}{x} \frac{dx}{\sinh 2qx} \{\sinh x (q + \iota\alpha) + \sinh x (q - \iota\alpha)\} \quad (58)$$

in search of alternatives.

It is readily seen that only one alternative is presented. If we notice that

$$2q-(q+\iota\alpha)=q-\iota\alpha, \qquad 2q-(q-\iota\alpha)=q+\iota\alpha$$

we may transform the equation into the equivalent form

$$\sum_0^\infty a_{2n}\{q_{2n}(q+\iota\alpha)+q_{2n}(q-\iota\alpha)\}$$
$$=\int_0^\infty \frac{\sin x}{x}\frac{dx}{\sinh 2qx}\{\sinh\{2q-(q+\iota\alpha)x\}+\sinh\{2q-(q-\iota\alpha)x\} \quad . \quad (59)$$

which is satisfied identically if, for suitable values of ρ,

$$\sum_0^\infty a_{2n}\,q_{2n}(\rho)=\int_0^\infty \frac{\sin x}{x}\frac{dx}{\sinh 2qx}\sinh(2q-\rho)\,x \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

At this point, if this alternative, which is unique in our method, were to fail, the method would also fail. Its success, now to be shown, implicitly carries with it the uniqueness of the resulting solution, which of course is necessary on physical grounds, as the problem in this aspect presupposes no further conditions to be satisfied. The limiting values of the coefficient a_{2n} provide the final test for any solution.

Again quoting POISSON's integrals in different notation,

$$\int_0^\infty \frac{\sinh x\,(2q-\rho)}{\sinh 2qx}\cos kx\,dx=\frac{\pi}{4q}\cdot\frac{\sin\frac{\pi}{2q}(2q-\rho)}{\cosh\frac{k\pi}{2q}+\cos\frac{\pi}{2q}(2q-\rho)}$$
$$=\frac{\pi}{4q}\sin\frac{\rho\pi}{2q}\Big/\left\{\cosh\frac{k\pi}{2q}-\cos\frac{\rho\pi}{2q}\right\}$$

we find by an easy integration with respect to k,

$$\int_0^\infty \frac{\sin x}{x}\frac{\sinh x\,(2q-\rho)}{\sinh 2qx}\,dx=\frac{\pi}{4q}\sin\frac{\rho\pi}{2q}\int_0^1\frac{dk}{\cosh\frac{k\pi}{2q}-\cos\frac{\rho\pi}{2q}}.$$

Using a new variable $t=\tanh\dfrac{k\pi}{2q}$,

this reduces to

$$\left[\tan^{-1}\left(\tanh\frac{k\pi}{4q}\cot\frac{\rho\pi}{4q}\right)\right]_0^1=\cot^{-1}\left(\tan\frac{\rho\pi}{4q}\Big/\tanh\frac{\pi}{4q}\right).$$

The last is a q-function, and our equation becomes

$$\sum_0^\infty a_{2n}q_{2n}(\rho)=q_0\left(\tan\frac{\rho\pi}{4q}\Big/\tanh\frac{\pi}{4q}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

When $q\to\infty$, the expression on the right becomes merely $q_0(\rho)$, and it is then evident by inspection that the limiting value of a_0 is unity, and of all other coefficients is zero, so that this mode of solution is satisfactory.

The problem, in this its last stage, is thus reduced to an expansion theorem in q-functions. Starting from a problem in these functions, our investigation, after passage through the integral equation, has returned to these functions with a problem of a very different though elegant type.

The procedure, according to which, by the action of a transcendental differential operation, cosh qx was at one point removed to the other side of an equation, may be regarded as objectionable. This difficulty can, however, be evaded by considering a more general equation, of which the one we discuss is a limiting case, and applying a method free from the use of differential operations. In a later investigation, described as the *Condenser problem*, we have a similar investigation to perform, and have therefore endeavoured to be brief. The following short justification of the procedure may be given :—

Introducing a parameter p, consider the equation

$$2\sum_0^\infty a_{2n}\int_0^\infty e^{-qx}J_{2n+\frac{1}{2}}(x)\frac{dx}{\sqrt{x}}\cosh px\cos\alpha x = \sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\sin x}{x}\cos\alpha x\frac{\cosh px}{\cosh qx}dx \quad (62)$$

where the case $p = q$ is that arising in our problem. If p never exceeds q, and if we can solve this equation in such a manner that a_{2n} is not a function of p, passage to the limit $p = q$ is clearly lawful, and our previous process is justified. This can be demonstrated readily. For the equation may be written in the form

$$\sum_0^\infty a_{2n}\int_0^\infty e^{-qx}J_{2n+\frac{1}{2}}(x)\frac{dx}{\sqrt{x}}\{\cosh x(p+\iota\alpha)+\cosh x(p-\iota\alpha)\}$$
$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\sin x}{x}\frac{dx}{\cosh qx}\{\cosh x(p+\iota\alpha)+\cosh x(p-\iota\alpha)\},$$

and if a_{2n} is not a function of p, p is always added to $\pm\iota\alpha$ wherever this occurs. Write

$$\sum_0^\infty a_{2n}\int_0^\infty e^{-qx}J_{2n+\frac{1}{2}}(x)\cosh\beta x\frac{dx}{\sqrt{x}} = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\sin x}{x}\frac{\cosh\beta x}{\cosh qx}dx$$

for all values of β whose real part does not exceed q, and the modulus of whose imaginary part does not exceed unity, and, moreover, with q_{2n} not dependent on β in these circumstances. Then the equation is necessarily satisfied. But this simple equation containing β may be written as

$$\sqrt{\frac{\pi}{2}}\sum_0^\infty a_{2n}\int_0^\infty J_{2n+\frac{1}{2}}(x)\frac{dx}{\sqrt{x}}\{e^{-x(q+\beta)}+e^{-x(q-\beta)}\}$$
$$= \int_0^\infty\frac{\sin x}{x}\frac{\cosh\beta x}{\cosh qx}dx$$
$$= \int_0^\infty\frac{\sin x}{x}\frac{dx}{\sinh 2qx}\{\sinh(q+\beta)x+\sinh(q-\beta)x\},$$

and is true for every value of β, with a_n not dependent on β, if such coefficients a_n can be found to satisfy

$$\sum_0^\infty a_{2n}\{q_{2n}(q+\beta)+q_{2n}(q-\beta)\}$$
$$=\int_0^\infty \frac{\sin x}{x}\cdot\frac{dx}{\sinh 2qx}\{\sinh(q+\beta)x+\sinh(q-\beta)x\} \quad . \quad (63)$$

which is our previous form with $\beta=\iota\alpha$, and, with no dubious intermediary stage. It leads in the same way to coefficients a_{2n} satisfying

$$\sum_0^\infty a_{2n}\,q_{2n}(\rho)=q_0\left(\tan\frac{\pi\rho}{4q}\Big/\tanh\frac{\pi}{4q}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

and we shall regard this as established.

§ 17. *Introduction of the Zonal Function* $Q_{2n}(\rho)$.

Beyond certain expansions in Q-functions given in O.S.H.* little is known regarding such series, and no general method is at hand. It is thus better to work with the zonal functions Q.

Their relations to the q-functions may be stated as follows† :—

Firstly,

$$q_{2n}(\iota\rho)=\iota^{-2n-1}Q_{2n}(\rho) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

where $Q_{2n}(\rho)$ is sufficiently defined, for argument greater than unity, by

$$Q_{2n}(\rho)=\frac{2^{2n}(2n\,!)^2}{(4n+1)\,!}\left\{\rho^{-2n-1}-\frac{(2n+1)(2n+2)}{2(4n+3)}\rho^{-2n-3}+\ldots\right\} \quad . \quad . \quad . \quad (66)$$

and also by

$$Q_{2n}(\rho)=\tfrac{1}{2}P_{2n}(\rho)\log\frac{\rho-1}{\rho+1}-\frac{4n-1}{1\,.\,2n}P_{2n-1}(\rho)-\frac{4n-5}{3(2n-1)}P_{2n-3}(\rho) \quad . \quad . \quad (67)$$

in a terminating form. The transformation we need, however, is for ρ less than or equal to unity, and requires some care. When $|\mu|<1$, $Q_{2n}(\mu)$ is usually defined by

$$Q_{2n}(\mu)=\tfrac{1}{2}P_{2n}(\mu)\log\frac{1+\mu}{1-\mu}-\frac{4n-1}{1\cdot 2n}P_{2n-1}(\mu)-\ldots \quad . \quad . \quad . \quad . \quad (68)$$

in CHRISTOFFEL's form, and is not the analytical continuation of the previous function. For this makes

$$Q_{2n}(\iota\rho)=\iota^{2n-1}\left\{p_{2n}(\rho)\tan^{-1}\rho+\frac{4n-1}{1\,.\,2n}p_{2n-1}(\rho)\ldots\right\},$$

leading to the conclusion

$$q_{2n}(\rho)=\frac{\pi}{2}P_{2n}(\iota\rho)+\iota\,Q_{2n}(\iota\rho) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

* Pp. 59–66.

† The classical memoir, in different notation, is that of Hobson, 'Phil. Trans.,' A, vol. 187, p. 443 (1896).

in terms of usual definitions of P and Q when $|\rho| < 1$, and of the definition of q we adopted after LAMB.*

Again, we can show in a similar manner that

$$q_{2n}(\iota\rho) = (-)^n \left\{\frac{\pi}{2} P_{2n}(\rho) - \iota Q_{2n}(\rho)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

and similarly

$$q_{2n}(-\iota\rho) = (-)^n \left\{\frac{\pi}{2} P_{2n}(\rho) + \iota Q_{2n}(\rho)\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

We have merely stated these results, which may easily be verified by the reader.

Thus the integral of the former paper,

$$\int_0^\infty e^{-qx} J_{2n+\frac{1}{2}}(x) \frac{dx}{\sqrt{x}} = \sqrt{\frac{2}{\pi}}\, q_{2n}(\rho),$$

leads to the conclusion that if α is real—writing $\iota\alpha$ for q—

$$\int_0^\infty \cos \alpha x\, J_{2n+\frac{1}{2}}(x) \frac{dx}{\sqrt{x}} = \tfrac{1}{2}\sqrt{\frac{2}{\pi}}\, \{q_{2n}(\iota\alpha) - q_{2n}(-\iota\alpha)\}$$

$$= (-)^n \sqrt{\frac{2}{\pi}}\, P_{2n}(\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

provided that α is less than unity. This has been proved otherwise by WEBER (and SCHAFHEITLIN), who has shown also that the integral is zero if $\alpha > 1$. On the other hand,

$$\int_0^\infty \sin \alpha x\, J_{2n+\frac{1}{2}}(x) \frac{dx}{\sqrt{x}} = \frac{1}{2\iota}\, \{q_{2n}(\iota\alpha) - q_{2n}(-\iota\alpha)\}$$

$$= (-)^n \sqrt{\frac{2}{\pi}}\, Q_{2n}(\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

with a Q instead of a P function. This has also been given by WEBER, and, unlike the other, is not restricted to the case $\alpha > 1$, though our proof is only applicable to this case. These integrals serve as verifications of the somewhat troublesome intricacies of the relations between q, Q, P when $\rho < 1$.

If now for small enough values of $|\rho|$ it is possible to write

$$f(\rho) = \sum_0^\infty A_{2n} q_{2n}(\rho) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

$f(\rho)$ being thus an odd function, we have also

$$f(\iota\rho) = \sum_0^\infty A_{2n} (-)^n \left\{\frac{\pi}{2} P_{2n}(\rho) - \iota Q_{2n}(\rho)\right\}$$

* "O.S.H.," p. 51.

where Q is the CHRISTOFFEL form. If $f(\iota\rho)$ has no real part, this involves, for $\rho < 1$, separately

$$\sum_0^\infty A_{2n}(-)^n P_{2n}(\rho) = 0, \qquad (75)$$

$$\sum_0^\infty A_{2n}(-)^n Q_{2n}(\rho) = \iota f(\iota\rho). \qquad (76)$$

Illustrations of this double relation between P and Q series with identical coefficients can be derived from some of the functions expanded in q-series in the earlier memoir, though the functions there discussed are "regular" in a sense not belonging to those with which we are now concerned. In the present problems, a function $f(\rho)$, odd for real values, becomes mixed when $\iota\rho$ is the variable, and the possibility of an expansion of its odd portion in Q-functions is not evident without some demonstration, which is given in due course for the function of our fundamental electrostatic problem. The demonstration is, of course, to the effect that the odd and even portions separately give rise to series 1) in Q's and 2) in P's whose coefficients differ only by a constant numerical factor—usually $\iota\pi/2$. This is a severe restriction on the types of function f which can satisfy the conditions of the problem, though we know, from the fact that the electrical problems must have solutions, that these functions must exist.

§ 18. *Expansions in Q and q Functions, Argument Less than Unity.*

We have seen that a real function $f(\rho)$ admitting

$$f(\rho) = \sum_0^\infty A_{2n} q_{2n}(\rho)$$

must also, if positive values of ρ are contemplated, admit also

$$f(\iota\rho) = \sum_0^\infty A_{2n}(-)^n \left\{\frac{\pi}{2} P_{2n}(\rho) - \iota Q_{2n}(\rho)\right\}$$

where $\rho < 1$, and therefore if

$$f(\iota\rho) = F_1(\rho) + \iota F_2(\rho) \qquad (77)$$

we have

$$\left.\begin{aligned} F_1(\rho) &= \frac{\pi}{2}\sum_0^\infty A_{2n}(-)^n P_{2n}(\rho) \\ F_2(\rho) &= -\sum_0^\infty A_{2n}(-)^n Q_{2n}(\rho) \end{aligned}\right\} \qquad (78)$$

Now the principal value of the integral

$$\int_{-1}^{1} \frac{P_{2n}(y)}{\rho - y}\, dy$$

is $Q_{2n}(\rho)$ when ρ is between ± 1 and the above form of $Q_{2n}(\rho)$ is used. If $F_1(\rho)$ is given for this range of ρ by

$$F_1(\rho) = \frac{\pi}{2}\sum_0^\infty A_{2n}(-)^n P_{2n}(\rho)$$

then

$$-\frac{2}{\pi}F_2(\rho) = \frac{\pi}{2}\sum_0^\infty A_{2n}(-)^n Q_{2n}(\rho)$$

$$= \frac{\pi}{4}\sum_0^\infty A_{2n}(-)^n \int_{-1}^1 \frac{P_{2n}(y)}{\rho - y}dy = \frac{1}{2}\int_{-1}^1 \frac{F_1(y)\,dy}{\rho - y}.$$

We do not need, for our present object, to examine the restrictions on this result. It is sufficient to notice that, in general, the functions admitting the two concurrent expansions are of the form

$$f(\iota\rho) = F_1(\rho) + \iota F_2(\rho)$$

$$= F_1(\rho) - \frac{\iota\pi}{4}\int_{-1}^1 \frac{F_1(y)\,dy}{\rho - y}$$

and that a consequent class of functions admitting the expansion

$$f(\rho) = \sum_0^\infty A_{2n}q_{2n}(\rho)$$

is

$$f(\rho) = F_1(-\iota\rho) - \frac{\iota\pi}{4}\int_{-1}^1 \frac{F_1(y)\,dy}{y + \iota\rho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

where F_1 is arbitrary.

We shall show that the function we require for electrostatics is one of this class, and that therefore a formal solution on the present lines is possible.

For if, when r is any integral index and ψ is arbitrary,

$$F_1(\rho) = P_r\{\psi(\rho)\}$$

and if

$$f(\iota\rho) = P_r\{\psi(\rho)\} - \frac{\iota\pi}{4}\int_{-1}^1 \frac{P_r(y)}{\psi(\rho) - y}dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

we have an obvious generalisation of f with the same properties. This function becomes

$$f(\iota\rho) = \frac{2}{\pi}\left\{\frac{\pi}{2}P_r\{\psi(\rho)\} - \iota Q_r\{\psi(\rho)\}\right\},$$

and when $|\rho| \not> 1$, must be expansible, with its odd and even parts separately in the two series. This condition, though necessary, is not sufficient, for the coefficients must, of course, in addition be definite and convergent.

Now the expansion we require to determine the coefficients a_{2n} is

$$\tan^{-1}(\tanh \pi/4q/\tan \pi\rho/4q) = \sum_0^\infty a_{2n}q_{2n}(\rho),$$

which transforms, for imaginary ρ, to

$$(-1<\rho<1)$$

$$\tan^{-1}(\tanh \pi/4q/\iota \tanh \pi\rho/4q) = \sum_0^\infty a_{2n}\rho_{2n}(\iota\rho)$$

$$= \sum_0^\infty a_{2n}(-)^n\left\{\frac{\pi}{2}\,\mathrm{P}_{2n}(\rho) - \iota\mathrm{Q}_{2n}(\rho)\right\}. \quad . \quad . \quad (81)$$

It remains to be shown that the left side is included in the above specification of type of function.

Since ρ is now real, and $|\rho| \leq 1$, the expression on the left is not purely imaginary, for the maximum of a hyperbolic tangent is unity. Write, where x is complex,

$$\tanh \pi/4q/\iota \tanh \rho\pi/4q = \tan x = -\iota\,(e^{2\iota x}-1)/(e^{2\iota x}+1),$$

where x is the value of the left-hand side. Then

$$e^{2\iota x} = -\left(\tanh\frac{\pi}{4q} + \tanh\frac{\rho\pi}{4q}\right)\Big/\left(\tanh\frac{\pi}{4q} - \tanh\frac{\rho\pi}{4q}\right),$$

and is essentially negative, and with the appropriate phase rule, we find

$$x = \tfrac{1}{2}\pi - \tfrac{1}{2}\iota \log\left\{\frac{1+\tanh\frac{\rho\pi}{4q}\Big/\tanh\frac{\pi}{4q}}{1-\tanh\frac{\rho\pi}{4q}\Big/\tanh\frac{\pi}{4q}}\right\},$$

which becomes

$$x = \tfrac{1}{2}\pi\mathrm{P}_0\left(\frac{\tanh \rho\pi/4q}{\tanh \pi/4q}\right) - \iota\mathrm{Q}_0\left(\frac{\tanh \rho\pi/4q}{\tanh \pi/4q}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

the function P_0 being, of course, unity. Our function is thus of the appropriate type, with

$$\psi(\rho) = \tanh \rho\pi/4q\Big/\tanh\frac{\pi}{4q},$$

and we have three distinct expansions involving the coefficients a_{2n}, ρ being in each case real, namely :—

$$\left.\begin{aligned} \tan^{-1}\psi(\rho) &= \sum_0^\infty a_{2n}\,q_{2n}(\rho) & & \\ \mathrm{Q}_0(\psi(\rho)) &= \sum_0^\infty (-)^n a_{2n}\mathrm{Q}_{2n}(\rho) & & (\rho<1) \\ \tfrac{1}{2} &= \sum_0^\infty (-)^n a_{2n}\mathrm{P}_{2n}(\rho) & & (\rho<1) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

which have been shown to be consistent. The third is not unique, in that many series of P functions have a constant sum. It is convenient to adhere to the second.

§ 19. *Final Solution for Two Parallel Circular Discs.*

We have discussed certain integral properties of the Q functions elsewhere.* The following results obtained in this paper may be quoted :—

$$Q_{2m}(\mu) = -\sum_{n=0}^{\infty} \frac{(4n+3)\,P_{2n+1}(\mu)}{(2m-2n-1)(2m+2n+2)},$$

$$\int_{-1}^{1} [Q_{2m}(\mu)]^2\,d\mu = 2\sum_{0}^{\infty} \frac{4n+3}{(2m-2n-1)^2(2m+2n+2)^2}$$

$$= \frac{2}{4m+1}\sum_{0}^{\infty} \frac{1}{(2m-2n-1)^2} - \frac{1}{(2m+2n+2)^2}$$

$$= \frac{2}{4m+1}\left\{\frac{\pi^2}{4} + \frac{d^2}{dz^2}\log\Gamma(z)\right\}_{z=2m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

in terms of gamma functions. They have been generalised further by WATSON. In particular,

$$\int_{-1}^{1} Q_0^2(\mu)\,d\mu = \pi^2/6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

We may now exhibit the final solution of our problem in an exact form. The function

$$\phi(x) = f(x)/\sqrt{x} = \sum_{0}^{\infty} \alpha_{2n}\,J_{2n+\frac{1}{2}}(x)/\sqrt{x}$$

has coefficients given by

$$Q_0(\tanh \pi\rho/4q/\tanh \pi/4q) = \sum_{0}^{\infty} (-)^n a_{2n}\,Q_{2n}(\rho) \qquad (\rho \leq 1)$$

and, accordingly,

$$(-)^n a_{2n}\int_{-1}^{1} [Q_{2n}(\rho)]^2\,d\rho = \int_{-1}^{1} Q_{2n}(\rho)\,Q_0\left(\frac{\tanh \pi\rho/4q}{\tanh \pi/4q}\right)d\rho. \quad . \quad . \quad . \quad (86)$$

The function $\phi(x)$ is thus now determined completely as a series of orthogonal functions. The coefficients a_{2n} are also, as we saw in an earlier section, those of the expansion of the potential of two equally charged and equally parallel discs in spheroidal harmonics.

Beyond the use of approximate methods, little can be done towards simplifying the integrals occurring in these coefficients, and such methods are not within the scope of this memoir, which is intended only to obtain the exact solutions.

For the purpose of obtaining $\phi(x)$ itself as a definite integral, we recall the formula of WEBER already mentioned, as

$$Q_{2n}(\rho) = (-)^n\sqrt{\frac{\pi}{2}}\int_{0}^{\infty} \sin \rho x\,J_{2n+\frac{1}{2}}(x)\,\frac{dx}{\sqrt{x}}$$

true for any value of ρ. Q, when $\rho \leq 1$, is the conventional Q. But when $\rho > 1$, it is $\iota^{-2n-1}\,q_{2n}(\iota\rho)$. Since when $\rho \leq 1$,

$$Q_0(\tanh \pi\rho/4q/\tanh \pi/4q) = \sum_{0}^{\infty} (-)^n a_{2n}Q_{2n}(\rho)$$

* 'Phil. Mag.,' March, 1923.

we have the same function equal to

$$\sqrt{\frac{\pi}{2}}\sum_0^\infty a_{2n}\int_0^\infty \sin \rho t\, J_{2n+\frac{1}{2}}(t)\frac{dt}{\sqrt{t}}.$$

A corresponding form is required when $\rho > 1$. Combining, then,

$$\sum_0^\infty a_{2n}\, q_{2n}(\iota\rho) = \tan^{-1}(\tanh \pi/4q/\tan \iota\rho\, \pi/4q)$$

or its equivalent, when $\rho > 1$

$$-\sum_0^\infty (-)^n a_{2n}\, Q_{2n}(\rho) = \tanh^{-1}(\tanh \pi/4q/\tanh \rho\pi/4q)$$

with WEBER's formula, we have, if $\rho < 1$

$$\tanh^{-1}(\tanh \pi/4q/\tanh \rho\pi/4q) = \sum_0^\infty a_{2n}\int_0^\infty \sin \rho t J_{2n+\frac{1}{2}}(t)\frac{dt}{\sqrt{t}}.$$

This and the corresponding form for $\rho > 1$ are equivalent to

$$\sqrt{\frac{\pi}{2}}\int_0^\infty \sin \rho t\phi(t)\,dt = \tanh^{-1}(\tanh \pi/4q/\tanh \rho\pi/4q)$$

when $\rho > 1$, and to the Q_0 function of the reciprocal argument when $\rho < 1$.

These formulæ can clearly be reversed by use of the Fourier double-integral theorem directly. Taking it in the form

$$\left.\begin{aligned}\psi(\rho) &= \int_0^\infty \sin \rho t\phi(t)\,dt,\\ \phi(x) &= \frac{2}{\pi}\int_0^\infty \sin \rho t\psi(t)\,dt\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

we have in the present case

$$\phi(x) = \left(\frac{2}{\pi}\right)^{3/2}\int_0^1 \sin xt\,dt\, Q_0(\tanh \pi t/4q/\tanh \pi/4q)$$
$$+\left(\frac{2}{\pi}\right)^{3/2}\int_1^\infty \sin xt\,dt\,\tanh^{-1}(\tanh \pi/4q/\tanh \pi t/4q) \quad (88)$$

giving $\phi(x)$ explicitly.

We shall not dwell further on the exact value of $\phi(x)$, as it is hardly capable of much simplification.

For physical application, we are more especially concerned with the first coefficient a_0 to the expansion of $\phi(x)$ in orthogonal functions. This is given by

$$\tfrac{1}{6}\pi^2 a_0 = a_0\int_{-1}^1 Q_0^2(\rho)\,d\rho = \int_{-1}^1 Q_0(\rho)\,Q_0(\tanh \pi\rho/4q/\tanh \pi/4q)\,d\rho. \quad . \quad . \quad (89)$$

§ 20. *Surface Density on the Discs.*

The potential produced by the discs at any point on the positive side of the z-axis is

$$V = a\sqrt{\frac{\pi}{2}}\int_0^\infty e^{-\lambda z}(1-e^{-\lambda c})\,J_0(\lambda\rho)\,\phi(a\lambda)\,d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

We may recall the positions of the discs, of radius a. One has its centre at $z=0$ and the other at $z=-c$, and their planes are perpendicular to z. They are maintained at equal potentials, and $q=2c/a$. The above value of ϕ being substituted, the integral to λ can be expressed, and V obtained as a single integral. But this integral is unwieldy, and it is better to note that the actual Bessel-Fourier solution is

$$V = \frac{a}{\pi}\int_0^\infty e^{-\lambda z}(1+e^{-\lambda c})\,J_0(\lambda\rho)\,d\lambda\int_0^1 \sin\lambda at\,dt Q_0(\tanh\pi at/2c/\tanh\pi a/2c)$$

$$+\int_1^\infty \sin\lambda at\,dt\,\tanh^{-1}(\tanh\pi a/2c/\tanh\pi at/2c). \quad . \quad . \quad . \quad (91)$$

This form alone shows that it could not have been built up by a process analogous to that of RIEMANN, in his investigation of Nobili's rings.

A consideration of the capacity of the discs when forming a condenser is more appropriately taken up later.

The potential being, when z is positive,

$$V = a\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\lambda z}(1+e^{-\lambda c})\,J_0(\lambda\rho)\,\phi(a\lambda)\,d\lambda \quad . \quad . \quad . \quad . \quad . \quad (92)$$

and also continuous at $z=0$, we have in the region between the discs

$$V = a\sqrt{\frac{\pi}{2}}\int_0^\infty \{e^{\lambda z}+e^{-\lambda(c+z)}\}\,J_0(\lambda\rho)\,\phi(a\lambda)\,d\lambda \quad . \quad . \quad . \quad . \quad . \quad (93)$$

and beyond $z=-c$ in the negative direction,

$$V = a\sqrt{\frac{\pi}{2}}\int_0^\infty e^{\lambda z}(1+e^{\lambda c})\,J_0(\lambda\rho)\,\phi(a\lambda)\,d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

evidently making V the same at $z=-c$ and $z=0$. This is the complete specification.

The distribution of charge between the two sides of a disc is readily found. If σ_1 is the surface density on the outer side of either disc, it is given by

$$\sigma_1 = \pm\frac{1}{4\pi}\left(\frac{\partial V}{\partial z}\right)_{x=0}.$$

Thus at a point ρ on either disc

$$\sigma_1 = \frac{a}{4\pi}\sqrt{\frac{\pi}{2}}\int_0^\infty \lambda(1+e^{-\lambda c})\,J_0(\lambda\rho)\,\phi(a\lambda)\,d\lambda. \quad . \quad . \quad . \quad . \quad . \quad (95)$$

The surface density σ_2 on the inner sides is similarly

$$\sigma_2 = \frac{a}{4\pi}\left(\frac{\partial V}{\partial z}\right)_{z=-c}$$

or

$$\sigma_2 = \frac{a}{4\pi}\sqrt{\frac{\pi}{2}}\int_0^\infty \lambda\,(1-e^{-\lambda c})\,J_0(\lambda\rho)\,\phi\,(a\lambda)\,d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

An inspection of the forms of these expressions serves to show the tendency of the charge to concentrate on the outer surfaces as c tends to zero, for σ_2 ultimately vanishes when the discs are very close together.

Part III.—Systems of more than Two Discs, or of Unequal Discs.

§ 21. *Transformation of Harmonic Series for Unequal Spheroids.*

In order to obtain solutions of problems involving two—or more—parallel co-axial discs of *unequal* radii, or more generally, two or more unequal spheroids with congruent major axes, belonging to confocal systems with different line constants, it is necessary to obtain a formula by which to transform the product $P_n(\mu')\,q_n(\zeta')$ to a summation of the form

$$\Sigma\, a_n P_n(\mu)\, p_n(\zeta)$$

where (μ, ζ), (μ', ζ') are spheroidal co-ordinates with different origins ($z = 0$, $z = -c$) and different radii (a, b) of confocality. Two such transformations are, in fact, needed, according to whether the new origin is beyond or behind the old, along the positive direction of the z-axis. We may designate these the *forward* and *backward transformations* respectively. The results are generalisations of formulæ earlier in this memoir, and their derivation will be set out somewhat briefly, for no essentially new features are introduced.

If there are two systems of spheroidal co-ordinates (μ', ζ'), (μ, ζ), of origins O_1, O_2 at the points $z = -c$, $z = 0$ on the axis of z, and limiting line-constants (a, b) respectively, we have for any external point (z, ρ)

$$\rho = a\sqrt{\{(1-\mu'^2)(1+\zeta'^2\}} = b\sqrt{\{(1-\mu^2)(1+\zeta^2)\}},$$
$$z = b\mu\zeta = -c + a\mu'\zeta',$$

and thus

$$\pi P_n(\mu')\,q_n(\zeta') = \int_0^\pi q_n\left[\mu'\zeta' - \iota\cos\phi\sqrt{\{(1-\mu'^2)(1-\zeta'^2)\}}\right]d\phi$$

$$= \int_0^\pi q_n\left\{\frac{c}{a} + \frac{b}{a}\mu\zeta - \iota\frac{b}{a}\sqrt{\{(1-\mu^2)(1+\zeta^2)\}}\cos\phi\right\}d\phi.$$

If

$$\varepsilon = \frac{b}{a}\mu\zeta - \iota\frac{b}{a}\cos\phi\sqrt{\{(1-\mu^2)(1+\zeta^2)\}},$$

then

$$\pi P_n(\mu')\,q_n(\zeta') = \int_0^\pi q_n\left(\frac{c}{a}+\varepsilon\right)d\phi.$$

Expanding q_n as before in a series of inverse powers, we can readily, as before, obtain the formula

$$\pi P_n(\mu') q_n(\zeta') = \frac{(-)^n 2^{2n+\frac{1}{2}} n!}{(2n+1)!} \Gamma(n+\tfrac{3}{2}) \frac{J_{n+\frac{1}{2}}(D)}{\sqrt{D}} \int_0^\pi \frac{d\phi}{\varepsilon + c/a}$$

where D is the operation $a\partial/\partial c$.

But

$$\int_0^\pi \frac{d\phi}{\varepsilon + c/a} = \int_0^\pi \frac{a\, d\phi}{c + b\mu\zeta - c\cos\phi \sqrt{\{(1-\mu^2)(1-\zeta^2)\}}}$$

$$= \frac{\pi a}{\sqrt{\{(c+b\mu\zeta)^2 + b^2(1-\mu^2)(1+\zeta^2)\}}} = \frac{\pi a}{O_1P}$$

where O_1P is the distance from P to the centre at $z = -c$.

By the inverse distance formula* for expansion into harmonics whose line constant or " radius " is b, this becomes identical with

$$\frac{\pi a}{b} \sum_{r=0}^{\infty} (-)^r (2r+1) P_r(\mu) p_r(\zeta) q_r\left(\frac{c}{b}\right).$$

Thus if (μ', ζ') relate to a spheroidal system of constant a at $z = -c$, and (μ, ζ) to another of constant b at $z = 0$,

$$P_n(\mu) q_n(\zeta') = (-)^n \sqrt{\frac{\pi}{2D}} \cdot J_{n+\frac{1}{2}}(D) \frac{a}{O_1P}$$

$$= (-)^n \sqrt{\frac{\pi}{2}} \cdot \frac{a}{b} \sum_0^{\infty} (-)^r (2r+1) P_r(\mu) p_r(\zeta) \frac{J_{n+\frac{1}{2}}(D)}{\sqrt{D}} q_r\left(\frac{c}{b}\right) \quad . \quad . \quad . \text{(I.) (97)}$$

where $D \equiv a/\partial/\partial c$.

This is only a slight generalisation of our previous formula, which ensues if $b = a$.

In dealing with problems involving two such spheroids—or parallel discs of equal radius—we require—even for a solution by successive approximations when c/a and c/b are large—transformations in *both* directions, so as to pass from one spheroid to the other and back again in the familiar zig-zag method of constant occurrence in the theory of image systems. We shall only obtain the fundamental formulæ useful at present, leaving others of similar type to be obtained by the reader. We shall call (I) the *forward* transformation, from an origin O_1 to an origin O_2 at distance c forward, along z increasing. The *backward* transformation, through a distance c along z decreasing, is the reverse of (I). For its determination, we may change from (μ, ζ) at the origin to (μ', ζ') at $(-c)$, according to the equations

$$b\mu\zeta = -c + a\mu'\zeta', \qquad b^2(1-\mu^2)(1+\zeta^2) = a^2(1-\mu'^2)(1+\zeta'^2),$$

the constants for (μ, ζ), (μ', ζ') being b and a respectively.

* " O.S.H.," p. 56.

Then

$$\pi P_n(\mu)\,q_n(\zeta) = \int_0^\pi q_n\,\{\mu\zeta - \iota\cos\phi\sqrt{\{(1-\mu^2)(1+\zeta^2)\}}\cos\phi\}\,d\psi$$

$$= \int_0^\pi q_n\left\{-\frac{c}{b} + \frac{a}{b}\mu'\zeta' - \frac{\iota a}{b}\cos\phi\sqrt{\{(1-\mu'^2)(1+\zeta'^2)\}}\right\}d\phi$$

$$= \int_0^\pi q_n\left(\varepsilon - \frac{c}{b}\right)d\phi$$

where

$$\varepsilon = \frac{b}{a}\mu'\zeta' - \frac{\iota b}{a}\cos\phi\sqrt{\{(1-\mu'^2)(1-\zeta'^2)\}}.$$

This becomes readily

$$\pi P_n(\mu)\,q_n(\zeta) = \frac{2^n n!}{(2n+1)!}\,2^{n+\frac{1}{2}}\,\Gamma(n+\tfrac{3}{2})\,\frac{J_{n+\frac{1}{2}}(-b\,\partial/\partial c)}{\sqrt{(-b\,\partial/\partial c)}}\int_0^\pi \frac{d\phi}{\varepsilon - c/b}.$$

The sign of the last integral requires special care, being negative in this instance, and not positive as would be assumed in a derivation of the backward from the forward transformation by a mere interchange of a and b and a change of sign of c. Thus

$$\int_0^\pi \frac{d\phi}{\varepsilon - e/b} = \int_0^\pi \frac{b\,d\phi}{-c + a\pi'\zeta' - ia\cos\phi\sqrt{\{(1-\mu'^2)(1+\zeta'^2)\}}}$$

where, in the space between the spheroids, the magnitude $a\mu'\zeta' - c$ is essentially negative. Moreover, if α is negative,

$$\int_0^\pi \frac{d\phi}{\alpha - \iota\beta\cos\phi} = -\frac{\pi}{\sqrt{(\alpha^2+\beta^2)}}$$

and thus on reduction

$$\int_0^\pi \frac{d\phi}{\varepsilon - e/b} = \frac{-\pi b}{O_2\rho} = \frac{-\pi b}{a}\sum_0^\infty (-)^r(2r+1)\,P_r(\mu')\,p_r(\zeta')\,q_r\left(-\frac{c}{a}\right)$$

and, finally,

$$P_n(\mu)\,q_n(\zeta) = (-)^{n+1}\frac{b}{a}\sqrt{\frac{\pi}{2}}\sum_0^\infty (-)^r(2r+1)\,P_r(\mu')\,p_r(\zeta')\,\frac{J_{n+\frac{1}{2}}(-b\,\partial/\partial c)}{\sqrt{(-b\,\partial/\partial c)}}\,q_r\left(-\frac{c}{a}\right),$$

but since the negative signs may be removed readily, we have

$$P_n(\mu)\,q_n(\zeta) = \frac{b}{a}\sqrt{\frac{\pi}{2}}\sum_0^\infty (2r+1)\,P_r(\mu')\,p_r(\zeta')\,\frac{J_{n+\frac{1}{2}}(b\partial/\partial c)}{\sqrt{b\partial/\partial c}}\,q_r\left(\frac{c}{a}\right). \quad . \text{ (II) } (98)$$

This is the backward transformation.

Using now the formula of the earlier memoir*

$$q_r\left(\frac{c}{a}\right) = \sqrt{\left(\frac{\pi a}{2}\right)}\int_0^\infty e^{-\lambda c}\,J_{n+\frac{1}{2}}(\lambda a)\,\frac{d\lambda}{\sqrt{\lambda}},$$

* "O.S.H.," p. 56.

we find

$$\frac{J_{n+\frac{1}{2}}(b\partial/\partial c)}{\sqrt{(b\partial/\partial c)}}\, q_r\left(\frac{c}{a}\right) = (-)^n \sqrt{\left(\frac{\pi a}{2b}\right)} \int_0^\infty e^{-\lambda c} J_{r+\frac{1}{2}}(\lambda a)\, J_{n+\frac{1}{2}}(\lambda b)\, \frac{d\lambda}{\lambda},$$

and similarly

$$\frac{J_{n+\frac{1}{2}}(a\partial/\partial c)}{\sqrt{(a\partial/\partial c)}}\, q_r\left(\frac{c}{b}\right) = (-)^n \sqrt{\left(\frac{\pi b}{2a}\right)} \int_0^\infty e^{-\lambda c} J_{r+\frac{1}{2}}(\lambda b)\, J_{n+\frac{1}{2}}(\lambda a)\, \frac{d\lambda}{\lambda};$$

so that (I) and (II) become

$$\left.\begin{aligned} &P_n(\mu')\, q_n(\zeta') \\ &\quad = \frac{\pi}{2}\sqrt{\frac{a}{b}} \sum_{r=0}^{\infty} (-)^r (2n+1)\, P_r(\mu)\, p_r(\zeta) \int_0^\infty e^{-\lambda c} J_{r+\frac{1}{2}}(\lambda b)\, J_{n+\frac{1}{2}}(\lambda a)\, \frac{d\lambda}{\lambda} \\ &P_n(\mu)\, q_n(\zeta) \\ &\quad = \frac{\pi}{2}(-)^n \sqrt{\frac{b}{a}} \sum_{r=0}^{\infty} (2r+1)\, P_r(\mu')\, p_r(\zeta') \int_0^\infty e^{-\lambda c} J_{r+\frac{1}{2}}(\lambda a)\, J_{n+\frac{1}{2}}(\lambda b)\, \frac{d\lambda}{\lambda} \end{aligned}\right\} \quad . \quad . \text{(III)} \quad (99)$$

The first may be used when ζ is small enough, and the second when ζ' is small enough—actually, though no proof is given, both are valid when the spheroids do not intersect or enclose one another. They may even touch.

It is convenient to have a set of symbols to represent the integrals in III. We shall write

$$\begin{Bmatrix} a & b & \\ n & r & c \end{Bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

for the first, so that the second becomes

$$\begin{Bmatrix} a & b & \\ r & n & c \end{Bmatrix}.$$

They are naturally simple generalisations of the function $K_r^n(\lambda)$ previously investigated, but lacking the commutative property

$$K_r^n(\lambda) = K_n^r(\lambda).$$

We shall not examine their analytical properties.

§ 22. *Two Unequal Parallel Co-axial Discs at Constant Potentials.*

We are now in a position to discuss this more general problem, and, as a preliminary, we may suppose the discs to be spheroids of centres (O_1, O_2), parameters (ζ_0', ζ_0) respectively, and maintained at any constant potentials (V_1, V_2).

The potential at any point may be taken to be

$$V = \sum_0^\infty a_n P_n(\mu')\, q_n(\zeta') + \sum_0^\infty b_n P_n(\mu)\, q_n(\zeta). \quad . \quad . \quad . \quad . \quad . \quad (101)$$

Applying the forward transformation, on the spheroid $\zeta = \zeta_0$ its value becomes

$$V = \sum_0^\infty b_r P_r(\mu)\, q_r(\zeta_0) + \frac{\pi}{2}\sqrt{\frac{a}{b}} \sum_0^\infty (-)^r (2r+1) P_r(\mu)\, p_r(\zeta_0) \sum_{n=0}^\infty a_n \begin{Bmatrix} b & a & \\ r & n & c \end{Bmatrix}.$$

This must be constant and equal to V_2. Thus we choose

$$\left.\begin{aligned} b_0 q_0(\zeta_0) + \frac{\pi}{2}\sqrt{\frac{a}{b}}\, p_0(\zeta_0) \sum_0^\infty a_n \begin{Bmatrix} b & a & \\ o & n & c \end{Bmatrix} &= V_2 \\ b_r q_r(\zeta_0) + \frac{\pi}{2}\sqrt{\frac{a}{b}} (-)^r (2r+1)\, p_r(\zeta_0) \sum_0^\infty a_n \begin{Bmatrix} b & a & \\ r & n & c \end{Bmatrix} &= 0 \; (r = 1, 2, \ldots) \end{aligned}\right\} \quad . \quad . \quad (102)$$

Applying now the backward transformation, the potential at any point on the other spheroid $\zeta' = \zeta_0'$ becomes

$$V = \sum_0^\infty a_r P_r(\mu')\, q_r(\zeta_0') + \frac{\pi}{2}\sqrt{\frac{a}{b}} \sum_{r=0}^\infty (2r+1)\, p_r(\zeta_0')\, P_r(\mu') \sum_{r=0}^\infty b_n (-)^n \begin{Bmatrix} b & a & \\ n & r & c \end{Bmatrix}$$

which must be constant and equal to V_1. Thus we may also write

$$\left.\begin{aligned} a_0 q_0(\zeta_0') + \frac{\pi}{2}\sqrt{\frac{b}{a}}\, p_0(\zeta_0') \sum_0^\infty (-)^n b_n \begin{Bmatrix} b & a & \\ n & o & c \end{Bmatrix} & V_1 \\ a_r q_r(\zeta_0') + \frac{\pi}{2}\sqrt{\frac{b}{a}} (2r+1)\, p_r(\zeta_0') \sum_0^\infty (-)^n b_n \begin{Bmatrix} b & a & \\ n & r & c \end{Bmatrix} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (103)$$

These four relations serve readily to determine, by a rapid approximation, the leading terms in the potential due to the spheroids when the central distance c is large. We pass, however, at once to the special case of discs of radii a and b, with a view to an exact reduction to integral equations.

In this case,

$$\zeta_0 = \zeta_0' = 0, \qquad p_{2n+1}(0) = 0$$

and we see at once that the odd coefficients a_{2n+1}, b_{2n+1} vanish in the expression for the potential. Moreover,

$$p_{2n}(0)/q_{2n}(0) = 2/\pi, \qquad 1/q_0(0) = 2/\pi$$

and the four formulæ become

$$b_0 = -\sqrt{\frac{a}{b}} \sum_0^\infty a_{2n} \begin{Bmatrix} b & a & \\ 0 & 2n & c \end{Bmatrix} + \frac{2V_2}{\pi},$$

$$a_0 = -\sqrt{\frac{b}{a}} \sum_0^\infty b_{2n} \begin{Bmatrix} b & a & \\ 2n & 0 & c \end{Bmatrix} + \frac{2V_1}{\pi},$$

$$b_{2r} = -(4r+1)\sqrt{\frac{a}{b}} \sum_0^\infty a_{2n} \begin{Bmatrix} b & a & \\ 2r & 2n & c \end{Bmatrix},$$

$$a_{2r} = -(4r+1)\sqrt{\frac{b}{a}} \sum_0^\infty b_{2n} \begin{Bmatrix} b & a & \\ 2n & 2r & c \end{Bmatrix}.$$

The potential is, on the positive side,

$$V = \sum_0^\infty \{a_{2n} P_{2n}(\mu') q_{2n}(\zeta') + b_{2n} P_{2n}(\mu) q_{2n}(\zeta)\} \qquad \text{(A)}$$

$$= \sqrt{\left(\frac{\pi a}{2}\right)} \int_0^\infty e^{-\lambda(z+c)} J_0(\lambda\rho) \frac{d\lambda}{\sqrt{\lambda}} \sum_0^\infty a_{2n} J_{2n+\frac{1}{2}}(\lambda a)$$

$$+ \sqrt{\left(\frac{\pi b}{2}\right)} \int_0^\infty e^{-\lambda z} J_0(\lambda\rho) \frac{d\lambda}{\sqrt{\lambda}} \sum_0^\infty b_{2n} J_{2n+\frac{1}{2}}(\lambda b)$$

or

$$V = \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-\lambda z} J_0(\lambda\rho) \frac{d\lambda}{\sqrt{\lambda}} \left\{ \sqrt{\frac{a}{b}} f(a\lambda) e^{-\lambda c} + \sqrt{\frac{b}{a}} F(b\lambda) \right\}$$

where the functions f and F are given by

$$f(ax) = \sqrt{b} \sum_0^\infty a_{2n} J_{2n+\frac{1}{2}}(ax)$$

$$F(bx) = \sqrt{a} \sum_0^\infty b_{2n} J_{2n+\frac{1}{2}}(bx).$$

The exact solution of the problem is thus reduced to a determination of these functions for substitution in (A), by means of the properties of the coefficients. We shall deduce for them two simultaneous integral equations analogous to that of the previous problem.

Two Simultaneous Integral Equations.—Using the integral expression

$$\begin{Bmatrix} b & a & c \\ 2r & 2n & \end{Bmatrix} = \int_0^\infty e^{-cx} J_{2r+\frac{1}{2}}(bx) J_{2n+\frac{1}{2}}(ax) \frac{dx}{x}$$

and its companion, with r and n interchanged, we find

$$a_{2r} = -(4r+1) \sqrt{\frac{b}{a}} \sum_0^\infty b_{2n} \int_0^\infty e^{-cx} J_{2n+\frac{1}{2}}(bx) J_{2r+\frac{1}{2}}(ax) \frac{dx}{x}$$

$$b_{2r} = -(4r+1) \sqrt{\frac{a}{b}} \sum_0^\infty a_{2n} \int_0^\infty e^{-cx} J_{2n+\frac{1}{2}}(bx) J_{2n+\frac{1}{2}}(ax) \frac{dx}{x},$$

except when $r = 0$, in which case we add, on the right, $2V_1/\pi$ and $2V_2/\pi$ respectively. These become

$$\left.\begin{aligned} a_{2r} &= -\sqrt{\frac{b}{a}} \int_0^\infty e^{-cx} \frac{dx}{x} \{(4r+1) J_{2r+\frac{1}{2}}(ax)\} \sum_0^\infty b_{2n} J_{2n+\frac{1}{2}}(bx) \\ &= -\sqrt{\frac{b}{a}} \int_0^\infty e^{-cx} \frac{dx}{x} \frac{F(bx)}{\sqrt{a}} (4r+1) J_{2r+\frac{1}{2}}(ax) \\ b_{2r} &= -\sqrt{\frac{a}{b}} \int_0^\infty e^{-cx} \frac{dx}{x} \frac{f(ax)}{\sqrt{b}} \,.\, (4r+1) J_{2r+\frac{1}{2}}(bx) \end{aligned}\right\} \quad . \quad . \quad (104)$$

respectively, with the necessary additions when $r = 0$. Let y be another variable independent of x—the general mode of procedure is the same as that adopted for the simpler problem of equal discs—and multiply the equations respectively by $J_{2r+\frac{1}{2}}(ay)$ and $J_{2r+\frac{1}{2}}(by)$, afterwards summing for all positive integral values of r: We thus find

$$\left.\begin{aligned}\sum_0^\infty a_{2r}J_{2r+\frac{1}{2}}(ay) &= \frac{2V_1}{\pi}J_{\frac{1}{2}}(ay) - \sqrt{\left(\frac{b}{a_2}\right)}\int_0^\infty e^{-cx}F(bx)\frac{dx}{x}S(a)\\ \sum_0^\infty b_{2r}J_{2r+\frac{1}{2}}(by) &= \frac{2V_2}{\pi}J_{\frac{1}{2}}(by) - \sqrt{\left(\frac{a}{b_2}\right)}\int_0^\infty e^{-cx}f(ax)\frac{dx}{x}S(b)\end{aligned}\right\} \quad . \quad . \quad (105)$$

where, when the variables are positive, and just excluding $x = y$,

$$\begin{aligned}S(\beta) &= \sum_0^\infty (4r+1)J_{2r+\frac{1}{2}}(\beta x)J_{2r+\frac{1}{2}}(\beta y)\\ &= \frac{1}{\pi}\sqrt{(xy)}\left\{\frac{\sin\beta(x+y)}{x+y} + \frac{\sin\beta(x-y)}{x-y}\right\},\end{aligned}$$

by a previously quoted summation. These forms are equivalent to

$$\left.\begin{aligned}\frac{f(ay)}{\sqrt{b}} - \frac{2V_1}{\pi}J_{\frac{1}{2}}(ay) &= -\frac{1}{\pi}\sqrt{\left(\frac{b}{a^2}\right)}\int_0^\infty e^{-cx}F(bx)\sqrt{(xy)}\frac{dx}{x}K_a(x,y)\\ \frac{F(by)}{\sqrt{a}} - \frac{2V_2}{\pi}J_{\frac{1}{2}}(by) &= -\frac{1}{\pi}\sqrt{\left(\frac{b_2}{a}\right)}\int_0^\infty e^{-cx}f(ax)\sqrt{(xy)}\frac{dx}{x}K_b(x,y)\end{aligned}\right\} \quad . \quad (106)$$

where

$$K_\beta(x,y) = \frac{\sin\beta(x+y)}{x+y} + \frac{\sin\beta(x-y)}{x-y} \quad . \quad . \quad . \quad . \quad . \quad . \quad (107)$$

and is a slight generalisation of the kernel of our previous integral equation, which was the case $\beta = 1$.

These are the fundamental integral equations on which the solution depends.

In the first instance, let $a = b$, so that the two discs are equal, but not of necessity equally charged. Then

$$\frac{f(ay)}{\sqrt{(ay)}} - \frac{2V_1}{\pi}\frac{J_{\frac{1}{2}}(ay)}{\sqrt{y}} = -\frac{1}{\pi}\int_0^\infty e^{-cx}F(ax)\frac{dx}{\sqrt{(ax)}}K_a(x,y),$$

$$\frac{F(ay)}{\sqrt{(ay)}} - \frac{2V_2}{\pi}\frac{J_{\frac{1}{2}}(ay)}{\sqrt{y}} = -\frac{1}{\pi}\int_0^\infty e^{-cx}f(ax)\frac{dx}{\sqrt{(ax)}}K_a(xy).$$

In one very important special case, $V_2 = -V_1$, where V_1 is the potential of the disc of centre $z = -c$. Then by addition,

$$f(ay) + F(ay) = -\frac{\sqrt{(ay)}}{\pi}\int_0^\infty e^{-cx}\frac{dx}{\sqrt{(ax)}}\{F(ax) + f(ax)\}K_a(x,y),$$

or, it is sufficient to take

$$F(ay) = -f(ay), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

and the equations become

$$\frac{f(ay)}{\sqrt{(ay)}} - \frac{2V_1}{\pi}\frac{J_{\frac{1}{2}}(ay)}{\sqrt{y}} = \frac{1}{\pi}\int_0^\infty e^{-cx} f(ax)\frac{dx}{\sqrt{(ax}} K_a(x, y). \quad . \quad . \quad . \quad . \qquad (109)$$

This is the *problem of the condenser* with equal and opposite charges on its plates.

In the case $V_2 = V_1$, the special case already discussed, we find in the same way, by subtraction, that $f(ay) = +F(ay)$, and

$$\frac{f(ay)}{\sqrt{(ay)}} - \frac{2V_1}{\pi}\frac{J_{\frac{1}{2}}(ay)}{\sqrt{y}} = -\frac{1}{\pi}\int_0^\infty e^{-cx} f(ax)\frac{dx}{\sqrt{(ax)}} K_a(x, y),$$

which is the same as

$$\frac{f(z)}{\sqrt{z}} - \frac{2V_1}{\pi}\sqrt{\left(\frac{2a}{\pi}\right)}\frac{\sin z}{z} = -\frac{1}{\pi}\int_0^\infty e^{cx/a} f(x)\frac{dx}{\sqrt{x}} K(x, y),$$

where $z = ay$. This is our previous form, and if

$$\frac{f(z)}{\sqrt{z}} = \frac{2V_1}{\pi} a^{\frac{1}{2}}\phi(z), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (110)$$

$\phi(z)$ is the function already determined in the previous problems. The potential is, however, now definite. We call this the *problem of equally charged plates*.

If as before

$$\phi(x) = \sum_0^\infty a_{2n} J_{2n+\frac{1}{2}}(x)/\sqrt{x}$$

the solution is, on the positive side,

$$V = \sqrt{\frac{\pi}{2}}\int_0^\infty e^{-\lambda z}(1 + e^{-\lambda c}) J_0(\lambda\rho) f(a\lambda)\frac{d\lambda}{\sqrt{\lambda}},$$

or

$$V = aV_1\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\lambda z} J_0(\lambda\rho)(1 + e^{-\lambda c})\phi(a\lambda)\,d\lambda. \quad . \quad . \quad . \quad . \quad . \qquad (111)$$

This completes the previous investigation. At a great distance since $\phi(a\lambda)$ is not oscillatory, we can take it as $a_0 J_{\frac{1}{2}}(a\lambda)/\sqrt{(a\lambda)}$ or $a_0(2/\pi)^{\frac{1}{2}}$, and V tends to become

$$V = \frac{2aV_1}{\pi}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)a_0$$

where (R_1, R_2) are distances from the discs. The charge on each disc is therefore

$$Q = \frac{2aV_1}{\pi}a_0 = \frac{12aV_1}{\pi^3}\int_0^1 Q_0(\rho)\,Q_0\left(\tanh\frac{\pi\rho}{4q}\Big/\tanh\frac{\pi}{4q}\right)d\rho \quad . \quad . \quad . \qquad (112)$$

(quoting the value of a_0) where q is $c/2a$. The charge on each disc required to give them unit potential is therefore

$$Q_1 = \frac{12a}{\pi^3}\int_0^1 Q_0(\rho)\, Q_0\left(\tanh\frac{\pi a\rho}{2c}\Big/\tanh\frac{\pi a}{2c}\right) d\rho. \quad . \quad . \quad . \quad . \quad (113)$$

When $c \to \infty$, the integral tends to

$$\int_0^1 [Q_0(\rho)]^2\, d\rho = \frac{\pi^2}{6},$$

and we obtain the usual capacity for a single disc.

In the problem of the condenser, if $z = ay$, the integral equation becomes

$$\phi(x) - \frac{J_{\frac{1}{2}}(z)}{\sqrt{z}} = \frac{1}{\pi}\int_0^\infty dx\phi(x)\, e^{-cx/a}\, K(x, z), \quad . \quad . \quad . \quad . \quad . \quad (114)$$

differing only from the other problem by containing the parameter $1/\pi$ instead of $-1/\pi$ in the other. If this equation is solved, functions f and F are

$$f(a\lambda) = -F(a\lambda)\,.\, \frac{2aV_1}{\pi}\sqrt{\lambda}\,.\,\phi(a\lambda),$$

and the potential on the positive side is

$$V = aV_1\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\lambda z}(e^{-\lambda c} - 1)\, J_0(\lambda\rho)\,\phi(a\lambda)\, d\lambda. \quad . \quad . \quad . \quad . \quad (115)$$

We shall return to this problem later.

§ 23. *The General Problem of Two Equal Discs.*

Whatever the potentials of the discs, the solution of this problem can be found from those of the two fundamental cases outlined above, one of which is already completed. For adding and subtracting the two integral equations with $a = b$, we find

$$\left.\begin{aligned} \frac{f(ay) + F(ay)}{\sqrt{(ay)}} - \frac{2}{\pi\sqrt{y}}(V_1 + V_2)\, J_{\frac{1}{2}}(ay) &= -\frac{1}{\pi}\int_0^\infty e^{-cx}\, dx\, \frac{\{F(ax) + f(ax)\}}{\sqrt{(ax)}}\, K_a(x, y) \\ \frac{f(ay) - F(ay)}{\sqrt{(ay)}} - \frac{2}{\pi\sqrt{y}}(V_1 - V_2)\, J_{\frac{1}{2}}(ay) &= +\frac{1}{\pi}\int_0^\infty e^{-cx}\, dx\frac{f(ax) + F(ax)}{\sqrt{(ax)}}\, K_a(x, y), \end{aligned}\right\} (116)$$

we thus have the functions $f + F$, $f - F$ satisfying respectively (1) the conditions for the equal parallel plates at potential $V_1 - V_2$, and (2) those for the condenser problem with potential $V_2 - V_1$ at $z = 0$, and $V_1 - V_2$ at $z = -c$.

If, therefore, we now solve the condenser problem, the general problem of parallel equal conducting plates is completed.

§ 24. *The Condenser Problem.*

In order to solve the equation

$$\phi(z) - J_{\frac{1}{2}}(z)/\sqrt{z} = \frac{1}{\pi}\int_0^\infty dx\, e^{-2qx}\phi(x)\,K(x,z) \qquad q = \frac{c}{a}$$

we notice that, as in the earlier investigation—to which this discussion is very similar, without the necessity of digressions in view of preceding work—the equation

$$\frac{1}{\pi}\int_0^\infty \left\{\phi(x) - \frac{J_{\frac{1}{2}}(x)}{\sqrt{x}} - e^{-2qx}\phi(x)\right\} K(x,z)\,dx = 0 \quad . \quad . \quad . \quad . \quad (117)$$

is implied. Or if

$$\phi(x) = \frac{(2\pi)^{-\frac{1}{2}}}{\sinh qx}\,Q(x)$$

then, by a similar argument to the foregoing,

$$\int_0^\infty \left\{e^{-qx}\,Q(x) - \frac{\sin x}{x}\right\} K(x,z)\,dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (118)$$

which is the *cardinal equation* of this problem. Its solution is now, however, to be an *odd* function Q (x). This will not constitute a solution unless it makes $\phi(x)$ expansible in the form

$$\phi(x) = \sum_0^\infty a_{2n}\,J_{2n+\frac{1}{2}}(x)/\sqrt{x}$$

and, this being the case, the coefficients a_{2n}, for all values of y, are given by

$$(2\pi)^{\frac{1}{2}}\sum_0^\infty a_{2n}\int_0^\infty e^{-qx}\sinh qx\,J_{2n+\frac{1}{2}}(x)\,K(x,y)\frac{dx}{\sqrt{x}} = \int_0^\infty \sin x K(x,y)\frac{dx}{x}. \quad . \quad (119)$$

The reader who refers to our previous discussion of the earlier integral equation can supply the intervening steps. The equivalent of this formula is

$$(2\pi)^{\frac{1}{2}}\sum_0^\infty a_{2n}\int_0^\infty e^{-qx}\sinh qx J_{2n+\frac{1}{2}}(x)\cos\alpha x\frac{dx}{\sqrt{x}} = \int_0^\infty \sin x\cos\alpha x\frac{dx}{x}, \quad . \quad . \quad (120)$$

where α must be capable of any value less than or equal to unity. Here we have, as before, replaced K (x, y) by an integral, and the coefficients a_{2n} are to be independent of α.

This equation only differs superficially from a corresponding one arising in the other problem by a change of cosh $q\,x$ to sinh $q\,x$. But this simple change makes it impossible to follow quite the same mode of solution, because the integral

$$\int_0^\infty \sin x\frac{\cos\alpha x}{\sinh qx}\frac{dx}{x}$$

does not converge. We cannot, therefore, get sinh qx to the other side of the equation directly, as we did with cosh qx previously. But sinh qx/x is an even function finite at $x = 0$, which can be so dealt with, and we may, therefore, seek a solution of

$$\sqrt{(2\pi)} \sum_0^\infty a_{2n} \int_0^\infty xe^{-qx} J_{2n+\frac{1}{2}}(x) \cos \alpha x \frac{dx}{\sqrt{x}} = \int_0^\infty \frac{\sin x}{\sinh qx} \cos \alpha x \, dx \quad (0 \leq \alpha \leq 1). \quad (121)\,(A)$$

The integral on the left is

$$-\frac{\partial}{\partial q} \int_0^\infty e^{-qx} J_{2n+\frac{1}{2}}(x) \cos \alpha x \frac{dx}{\sqrt{x}} = -\frac{1}{\sqrt{(2\pi)}} \{q_{2n}'(q + \iota\alpha) + q_{2n}'(q - \iota\alpha)\},$$

so that the equation becomes

$$-\sum_0^\infty a_{2n} \{q_{2n}'(q + \iota\alpha) + q_{2n}'(q - \iota\alpha)\} = \int_0^\infty \frac{\sin x}{\sinh qx} \cos \alpha x \, dx$$

$$= \int_0^\infty dx \frac{\sin x}{\sinh 2qx} \{\cosh(q + \iota\alpha) x + \cosh(q - \iota\alpha) x\}$$

by easy algebra. Here values of a_{2n} must be found independent of α, or the method fails. It is indicated that we should solve the simpler equation

$$-\sum_0^\infty a_{2n} q_{2n}'(q + \iota\alpha) = \int_0^\infty \frac{\sin x}{\sinh 2qx} \cosh(q + \iota\alpha) x \, dx \quad . \quad . \quad . \quad . \quad (122)$$

—a process which would be effective as a solution—or if $q + \iota\alpha = \rho$, and a_{2n} is independent of ρ,

$$\sum_0^\infty a_{2n} q_{2n}'(\rho) = -\int_0^\infty \frac{\sin x}{\sinh 2qx} \cos \lambda\rho x \, dx. \quad . \quad . \quad . \quad . \quad . \quad (123)$$

But it can be shown that although this leads, of necessity, to a solution, it is not the solution we seek. (It has hydrodynamical applications not relevant in this memoir.) The final test of any solution is as before, when $q \to \infty$,

$$a_0 \to 1, \qquad a_n \to 0, \qquad (n \neq 0)$$

This was effective in the problem of two equally charged discs, and is equally able to discriminate, and pick out the unique solution, in this case also.

Without further discussion of other artifices which may be suggested, and which yield solutions not germane to our present object, we pass at once to the appropriate solution, for brevity.

We can write the equation

$$-\sum_0^\infty a_{2n} \{q_{2n}'(q + \iota\alpha) + q_{2n}'(q - \iota\alpha)\} = \int_0^\infty \frac{\sin x}{\sinh qx} \cos \alpha x \, dx$$

in the form

$$-\sum_0^\infty a_{2n} \{q_{2n}'(q + \iota\alpha) + q_{2n}'(q - \iota\alpha)\}$$

$$= \int_0^\infty \frac{\sin x}{2 \sinh^2 qx} dx \{\sinh[2q - (q + \iota\alpha)] x + \sinh[2q - (q - \iota\alpha)] x\}$$

by simple algebra, and a solution is found by selecting the coefficients a_{2n} to satisfy, for all values of a variable ρ,

$$-\sum_0^\infty a_{2n} q_{2n}'(\rho) = \int_0^\infty \frac{\sin x}{\sinh qx} \sinh(2q-\rho)\, x\, dx. \quad . \quad . \quad . \quad (124)\ (B)$$

This will appear later to be the proper artifice in this case. Before we proceed further, however, we notice an important respect in which our procedure with this equation is open to serious objection. For the integral

$$\int_0^\infty \sin x \cos \alpha x \frac{dx}{x}$$

is not of a type to which a differentiation of the form $D/\sin qD$, $(D \equiv \partial/\partial\alpha)$ can be applied obviously, as in the manner of transferring $\sinh qx$ to the other side of (A). We can, however, evade this difficulty by introducing a parameter p, not greater than q, and considering, instead of (A), the equation

$$\sqrt{(2\pi)} \sum_0^\infty a_{2n} \int_0^\infty e^{-qx} \sinh px \, J_{2n+\frac{1}{2}}(x) \cos \alpha x \frac{dx}{\sqrt{x}} = \int_0^\infty \sin x \frac{\sinh px}{\sinh qx} \cos \alpha x \frac{dx}{x} \qquad (125)$$

as defining a_{2n} in a more general mathematical problem, for all values of p between zero and q. Then in the limit $p = q$ to which the analysis can be extended, we have

$$\sqrt{(2\pi)} \sum_0^\infty a_{2n} \int_0^\infty e^{-qx} \sinh qx \, J_{2n+\frac{1}{2}}(x) \cos \alpha x \frac{dx}{\sqrt{x}} = \int_0^\infty \sin x \cos \alpha x \frac{dx}{x},$$

which is equation (A). The more general equation becomes

$$\sqrt{(2\pi)} \sum_0^\infty a_{2n} \int_0^\infty \frac{dx}{\sqrt{x}} J_{2n+\frac{1}{2}}(x) \{e^{(p-q+\iota\alpha)x} + e^{(p-q-\iota\alpha)x} - e^{(-p-q+\iota\alpha)x} - e^{-(p-q-\iota\alpha)x}\}$$
$$= \tfrac{1}{2} \int_0^\infty \frac{\sin x\, dx}{x \sinh qx} \{\sinh(p+\iota\alpha)x + \sinh(p-\iota\alpha)x\}.$$

The left side becomes

$$\tfrac{1}{2} \sum_0^\infty a_{2n} \{q_{2n}(q-p-\iota\alpha) + q_{2n}(q-p+\iota\alpha) - q_{2n}(p+q-\iota\alpha) - q_{2n}(p+q+\iota\alpha)\}$$

and the right can be transformed to

$$\tfrac{1}{2} \int_0^\infty \frac{\sin x\, dx}{2x \sinh^2 qx} \{\cosh(q+p+\iota\alpha)x - \cosh(q-p-\iota\alpha)x + \cosh(q+p+\iota\alpha)x$$
$$- \cosh(q-p+\iota\alpha)x\}.$$

Differentiate with respect to α, and, for convenience, write

$$p + \iota\alpha = \rho_1, \qquad p - \iota\alpha = \rho_2.$$

There results the identity

$$\left.\begin{aligned} -\sum_0^\infty a_{2n}\{q_{2n}'(q-\rho_1)+q_{2n}'(q+\rho_1)-q_{2n}'(q-\rho_2)-q_{2n}'(q+\rho_2)\} \\ =\int_0^\infty \frac{\sin x\,dx}{2\sinh^2 qx}\{\sinh(q+\rho_1)x+\sinh(q-\rho_1)x-\sinh(q-\rho_2)x \\ -\sinh(q+\rho_2)x\}, \end{aligned}\right\} \quad (126)$$

which is evidently satisfied if, for all values of ρ,

$$-\sum_0^\infty a_{2n}\{q_{2n}'(q-\rho)+q_{2n}'(q+\rho)\}=\int_0^\infty \frac{\sin x\,dx}{2\sinh^2 qx}\{\sinh(q+\rho)x+\sinh(q-\rho)x\}. \quad (127)$$

This is the equation immediately preceding (B), from which the ensuing argument follows as before—with only $p+\iota\alpha$ for $\iota\alpha$, which is of no importance.

Thus our process is in fact valid for a much more general equation than that under review. We continue, therefore, with equation (B), of the form

$$-\sum_0^\infty a_{2n}q_{2n}'(\rho)=\int_0^\infty \frac{\sin x\,dx}{2\sinh^2 qx}\sinh(2q-\rho)x \quad . \quad . \quad . \quad . \quad . \quad (128)$$

defining the coefficients a_{2n}. The integral on the right can be evaluated, though tediously.

Let it be called I, and write $u=q-\rho$. Then

$$\begin{aligned} \mathrm{I} &= \int_0^\infty \frac{\sin x\,dx}{2\sinh^2 qx}(\sinh qx\cosh ux+\cosh ux\sinh ux) \\ &= \tfrac{1}{2}\int_0^\infty \frac{\sin x}{\sinh qx}\cosh ux\,dx+\tfrac{1}{2}\int_0^\infty \sin x\sinh ux\frac{\cosh qx}{\sinh^2 qx}dx \\ &= \tfrac{1}{2}\mathrm{I}_1-\tfrac{1}{2}\frac{d\mathrm{I}_2}{dq}, \end{aligned}$$

where

$$\mathrm{I}_2=\int_0^\infty \frac{\sin x\sinh ux}{x\sinh qx}dx, \qquad \mathrm{I}_1=\int_0^\infty \frac{\cosh ux}{\sinh qx}\sin x\,dx,$$

—we are assuming, as we may by the arbitrary character of ρ, that the relative values of ρ and q are such as to secure a finite set of integrals.

Now I_1 is one of POISSON'S integrals, given by

$$\mathrm{I}_1=\frac{\pi}{2q}\sinh\frac{\pi}{q}\Big/\left(\cosh\frac{\pi}{q}+\cos\frac{u\pi}{q}\right),$$

and I_2 is the integral of another with regard to a parameter, and quoting this,

$$\begin{aligned} \mathrm{I}_2 &= \int_0^1 d\lambda\int_0^\infty \frac{\sinh ux}{\sinh qx}\cos\lambda x\,dx \\ &= \frac{\pi}{2q}\sin\frac{u\pi}{q}\int_0^1\frac{d\lambda}{\cosh\lambda\pi/q+\cos u\pi/q} \\ &= \tan^{-1}\left(\tanh\frac{\pi}{2q}\tan\frac{u\pi}{2q}\right) \end{aligned}$$

after an easy reduction. Thus, finally,

$$-\sum_0^\infty a_{2n}q_{2n}'(\rho) = \frac{\pi}{4q}\cdot\frac{\sinh\pi/q}{\cosh\pi/q+\cos u\pi/q} - \tfrac{1}{2}\frac{d}{dq}\tan^{-1}\left(\tanh\frac{\pi}{2q}\tan\frac{u\pi}{2q}\right),$$

where, after the differentiation, u is equated to $q-\rho$. The final result is, after further reduction,

$$-\sum_0^\infty a_{2n}q_{2n}'(\rho) = \frac{\pi}{2q}\frac{\sinh\pi/q}{\cosh\pi/q-\cos\rho\pi/q} + \frac{\pi}{4q^2}\frac{\sin\rho\pi/q-\rho\sinh\pi/q}{\cosh\pi/q-\cos\rho\pi/q}. \quad . \quad . \quad (129)$$

When $\rho\to\infty$, this becomes

$$-\sum_0^\infty a_{2n}q_{2n}'(\rho) = -\frac{1}{1-\rho^2},$$

whence

$$\sum_0^\infty a_{2n}q_{2n}(\rho) = \cot^{-1}\rho = q_0(\rho),$$

from which

$$a_0 = 1, \qquad a_2 = a_4 = \ldots = 0.$$

The fundamental test of the electrostatic problem is therefore satisfied.

Nevertheless, the expression is not possible as it stands, for on the right side the first function is even in ρ and the second odd, while $q_{2n}(\rho)$ and therefore the left side, is always even. But reference to the original argument by which this function was found shows at once that it is equally valid with $-\rho$ for ρ. The proper form we seek—which still satisfies the condition when $\rho\to\infty$—is the mean of those from $+\rho$ and $-\rho$, and becomes

$$-\sum_0^\infty a_{2n}q_{2n}'(\rho) = \frac{\pi}{2q}\frac{\sinh\pi/q}{\cosh\pi/q-\cos\rho\pi/q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

which is a possible expansion.

This could have been deduced more readily by selecting a simpler effective artifice in the first place. But our present procedure is of value as an illustration of the type of analysis involved in solving the integral equation with prescribed conditions as $p\to\infty$. The integrated form of the last equation is readily found as

$$\sum_0^\infty a_{2n}q_{2n}(\rho) = \tan^{-1}\left(\frac{\tanh\pi/2q}{\tan\rho\pi/2q}\right) = q_0\left(\frac{\tan\rho\pi/2q}{\tanh\pi/2q}\right). \quad . \quad . \quad . \quad . \quad . \quad (131)$$

This may be compared with the corresponding formula for the coefficients a_{2n} in the problem of two equally charged plates—the only final difference is $2q$ for $4q$, all the later part of the solution of that problem being directly applicable with this simple alteration. We do not need, therefore, to repeat the arguments relating to the validity of the present expansion, and the mode by which the coefficients a_{2n} are determined from it. The general result with $2q$ for $4q$ will suffice, and we conclude that

$$(-)^n a_{2n}\int_{-1}^1 Q_{2n}^2(\rho)\,d\rho = \int_{-1}^1 Q_0(\rho)\,Q_0(\tanh\rho\pi/2q/\tanh\pi/2q)\,d\rho \quad . \quad . \quad (132)$$

and the general value of $\phi(x)$ in this problem is

$$\phi(n) = \sum_0^\infty a_{2n} J_{2n+\frac{1}{2}}(x)/\sqrt{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

in a series of orthogonal functions.

The a's are also the coefficients in the expansion of the potential at any point in spherical harmonics.

Also by analogy with the previous problem, the solution of the condenser problem with discs at potentials $\pm V$, and at distance c apart is, on the positive side,

$$V = a V_1 \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\lambda z} (e^{-\lambda c} - 1) J_0(\lambda\rho) \phi(a\lambda) \, d\lambda$$

where

$$\phi(a\lambda) = \sum_0^\infty (-)^n \frac{J_{2n+\frac{1}{2}}(a\lambda)}{\sqrt{(a\lambda)}} \int_0^1 Q_{2n}(t) Q_0\left(\frac{\tanh \pi at/c}{\tanh \pi a/c}\right) dt \Big/ \int_0^1 Q_{2n}{}^2(t) \, dt \quad . \quad . \quad (134)$$

§ 25. *Capacity of the Double Disc Condenser.*

At a very great distance, since λ is a function of a non-exponential type, and c is small in comparison with z, we may write, for the leading terms in R^{-1}, where R is the distance from the origin,

$$V = -a V_1 \sqrt{\frac{2}{\pi}} \int_0^\infty \{\phi(0) + a\lambda\phi'(0) + \ldots\} \left\{\lambda c - \frac{\lambda^2 c^2}{2!} \ldots\right\} e^{-\lambda z} J_0(\lambda\rho) \, d\lambda$$

and the first term is

$$V = -a V_1 c \sqrt{\frac{2}{\pi}} \phi(0) \int_0^\infty \lambda e^{-\lambda z} J_0(\lambda\rho) \, d\lambda$$

$$= a V_1 c \sqrt{\frac{2}{\pi}} \phi(0) \frac{\partial}{\partial z}\left(\frac{1}{R}\right),$$

terminating what is essentially a spherical harmonic series with its first term, which represents the effect of the condenser regarded as a simple doublet.

If the charges on the two discs are respectively $\pm Q$, we thus find

$$Q = a V_1 \sqrt{\frac{2}{\pi}} \phi(0)$$

where

$$\phi(0) = a_0 [J_{\frac{1}{2}}(x)/\sqrt{x}]_{x=0} = \sqrt{\frac{2}{\pi}} a_0$$

and quoting the value of a_0 in this case,

$$Q = \frac{12 a V_1}{\pi^3} \int_1^1 Q_0(t) Q_0(\tanh \pi at/c / \tanh \pi a/c) \, dt$$

relating charge and potential of either disc. The *capacity of the condenser* is $C = Q/2V_1$, or

$$C = \frac{6a}{\pi^3}\int_1^1 Q_0(t)\, Q_0(\tanh \pi at/c/\tanh \pi a/c)\, dt \quad . \quad . \quad . \quad . \quad . \quad (135)$$

reducing to the proper result when $c \to \infty$.

The integral cannot be evaluated in simple terms, but readily yields approximations when (1) c is large, (2) c is small.

§ 26. *Charged Disc in front of an Infinite Plane at Zero Potential.*

The two discs of the condenser evidently produce zero potential over the plane $z = -\frac{1}{2}c$ midway between them. The same solution is therefore valid for the problem of a disc in front of such a plane, provided that c now represents *twice* the distance of disc from plane. We can therefore use the capacity formula for a disc parallel to a much larger disc, with this understanding regarding c. The result is of special importance when c/a is small, if a is the radius of the smaller disc.

§ 27. *Coefficients of Capacity and Induction of Two Equal Parallel Discs.*

Reverting now to the general problem of two equal discs, that at $z = 0$ with potential V_2, and that at $z = -c$ with potential V_1, we recall the functions f, F, where the whole potential produced when z is positive becomes

$$V = \sqrt{\frac{\pi}{2}}\int_0^\infty e^{-\lambda z} J_0(\lambda\rho)\frac{d\lambda}{\sqrt{\lambda}}\{f(a\lambda)\, e^{-\lambda c} + F(a\lambda)\} \quad . \quad . \quad . \quad . \quad (136)$$

Moreover, in the condenser problem, with potential $-V_3$ on the right ($z = 0$) and $+V_3$ on the left,

$$f(a\lambda)/\sqrt{\lambda} = \frac{2a}{\pi} V_3\, \phi_1(a\lambda)$$

where ϕ_1 has just been determined, and in the two disc problem of equal potential V_3,

$$f(a\lambda)/\sqrt{\lambda} = \frac{2a}{\pi} V_3\, \phi_2(a\lambda)$$

ϕ_2 being the previous ϕ function.

These define f for the two cases. We have seen, moreover, that if (f, F) relate to the general case, $f + F$ is the function for the equally charged plates each of potential $V_1 + V_2 = V_3$, and $f - F$ is the function for the condenser with $V_1 - V_2 = V_3$. Accordingly

$$\left.\begin{aligned} f + F &= \frac{2a}{\pi}(V_1 + V_2)\,\phi_2(a\lambda) \\ f - F &= \frac{2a}{\pi}(V_1 - V_2)\,\phi_1(a\lambda) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (137)$$

whence

$$\left.\begin{aligned}\frac{\pi f}{a} &= V_1(\phi_1+\phi_2)+V_2(\phi_1-\phi_2)\\ \frac{\pi F}{a} &= V_1(\phi_1-\phi_2)+V_2(\phi_1+\phi_2)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

When the discs have potentials (V_1, V_2) they therefore produce an external potential on the right, equal to

$$V = \frac{a}{\sqrt{(2\pi)}}\int_0^\infty e^{-\lambda z} J_0(\lambda e)\frac{d\lambda}{\sqrt{\lambda}}\{e^{-\lambda c}V_1(\phi_1+\phi_2)+e^{-\lambda c}V_2(\phi_1-\phi_2) + V_1(\phi_1-\phi_2)+V_2(\phi_1+\phi_2)\}, \quad (139)$$

where $a\lambda$ is the argument of ϕ_1 and of ϕ_2, and

$$\phi_1(a\lambda) = \sum_0^\infty (-)^n \frac{J_{2n+\frac{1}{2}}(a\lambda)}{\sqrt{(a\lambda)}}\int_0^1 Q_{2n}(t)\,Q_0\left(\frac{\tanh \pi at/c}{\tanh \pi a/c}\right)dt \Big/ \int_0^1 Q_{2n}{}^2(t)\,dt, \quad . \quad (140)$$

and $\phi_2(a\lambda)$ is the same function with $2c$ for c.

Let

$$\phi_1(0) = (2/\pi)^{\frac{1}{2}} a_0, \qquad \phi_1(0) = (2/\pi)^{\frac{1}{2}} b_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

Then taking the leading terms in the integral, the potential at a great distance (R_1 and R_2 from the discs) is

$$V = \frac{a}{\pi R_2}\{V_2(b_0-a_0)+V_1(b_0+a_0)\} + \frac{a}{\pi R_1}\{V_1(b_0-a_0)+V_2(b_0+a_0)\},$$

whence the charges are

$$\left.\begin{aligned}Q_1 &= \frac{a}{\pi}\{V_1(b_0+a_0)+V_2(b_0-a_0)\}\\ Q_2 &= \frac{a}{\pi}\{V_2(b_0+a_0)+V_1(b_0-a_0)\}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (142)$$

The coefficients of capacity and induction are therefore known, for

$$a_0 = \frac{6}{\pi^2}\int_{-1}^1 Q_0(t)\,Q_0(\tanh \pi at/c/\tanh \pi a/c)\,dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

$$b_0 = \frac{6}{\pi^2}\int_{-1}^1 Q_0(t)\,Q_0(\tanh \pi at/2c/\tanh \pi a/2c)\,dt. \quad . \quad . \quad . \quad . \quad . \quad (144)$$

§ 28. *The Problem of Three Parallel Discs at Potentials* V_1, V_2, V_3.

We may finally indicate how the solution of these problems can proceed further, without giving the detailed analysis. Let the three discs have radii (α, β, γ), and have their centres at $z = -c_1$, $z = 0$, $z = +c_2$ respectively. Measured with reference to these centres, the spheroidal co-ordinates of an external point will be (μ', ζ'), (μ, ζ) (μ'', ζ'') respectively.

Let the potential at P be given by the formula, satisfying the conditions of the usual type—such as vanishing at infinity—

$$V = \sum_0^\infty \{a_n P_n(\mu') q_n(\zeta') + b_n P_n(\mu) q_n(\zeta) + c_n P_n(\mu'') q_n(\zeta''). \quad . \quad . \quad . \quad (145)$$

On the central disc, this becomes

$$V = \sum_0^\infty b_r P_r(\mu) q_r(\zeta) + \frac{\pi}{2} \sum_{r=0}^\infty (-)^r (2r+1) P_r(\mu) p_r(\zeta) \sum_{n=0}^\infty a_n \sqrt{\frac{\alpha}{\beta}} \begin{Bmatrix} \alpha & \beta & \\ n & r & c_1 \end{Bmatrix}$$
$$+ \frac{\pi}{2} \sum_{r=0}^\infty (2r+1) P_r(\mu) p_r(\zeta) \sum_{n=0}^\infty (-)^n c_n \sqrt{\frac{\gamma}{\beta}} \begin{Bmatrix} \beta & \gamma & \\ r & n & c_2 \end{Bmatrix}$$

where the forward transformation through a distance c_1 gives rise to one series and the backward transformation through c_2 to the other.

This must be equal to V_2 when $\zeta = 0$, whence since $p_{2r+1}(0) = 0$, we find $b_{2r+1} = 0$, and with $q_0(0) = \pi/2$, $p_{2r}(0)/q_{2r}(0) = 2/\pi$, we find further that

$$\left.\begin{aligned} & b_0 + \sqrt{\frac{\alpha}{\beta}} \sum_0^\infty a_n \begin{Bmatrix} \alpha & \beta & \\ n & 0 & c_1 \end{Bmatrix} + \sqrt{\frac{\gamma}{\beta}} \sum_0^\infty (-)^n c_n \begin{Bmatrix} \beta & \gamma & \\ 0 & n & c_2 \end{Bmatrix} = \frac{2}{\pi} V_2 \\ & b_{2r} + (4r+1) \sqrt{\frac{\alpha}{\beta}} \sum_0^\infty a_n \begin{Bmatrix} \alpha & \beta & \\ n & 2r & c_1 \end{Bmatrix} + (4r+1) \sqrt{\frac{\gamma}{\beta}} \sum_0^\infty (-)^n c_n \begin{Bmatrix} \beta & \gamma & \\ 2r & n & c_2 \end{Bmatrix} = 0 \\ & \qquad (r = 1, 2, 3 \ldots) \end{aligned}\right\} . \quad (146)$$

On the disc of potential V_1, using the backward transformation in one case through c_1, and, in the other, through $c_1 + c_2$, we find

$$V = \sum_0^\infty a_r P_r(\mu') q_r(\zeta') + \frac{\pi}{2} \sum_{r=0}^\infty (2r+1) P_r(\mu') p_r(\zeta') \sum_0^\infty (-)^n b_n \sqrt{\frac{\beta}{\alpha}} \begin{Bmatrix} \alpha & \beta & \\ r & n & c_1 \end{Bmatrix}$$
$$+ \frac{\pi}{2} \sum_{r=0}^\infty (2r+1) P_r(\mu') p_r(\zeta') \sum_{n=0}^\infty (-)^n c_n \begin{Bmatrix} \alpha & \gamma & \\ r & n & c_1 + c_2 \end{Bmatrix} \sqrt{\frac{\gamma}{\alpha}},$$

and equating this to V_1, when $\zeta' = 0$, we readily find

$$\left.\begin{aligned} & a_0 + \sqrt{\frac{\beta}{\alpha}} \sum_0^\infty (-)^n b_n \begin{Bmatrix} \alpha & \beta & \\ 0 & n & c_1 \end{Bmatrix} + \sqrt{\frac{\gamma}{\alpha}} \sum_0^\infty (-)^n c_n \begin{Bmatrix} \alpha & \gamma & \\ 0 & n & c_1 + c_2 \end{Bmatrix} = \frac{2}{\pi} V_1 \\ & a_{2r} + (4r+1) \sqrt{\frac{\beta}{\alpha}} \sum_0^\infty (-)^n b_n \begin{Bmatrix} \alpha & \beta & \\ 2r & n & c_1 \end{Bmatrix} \\ & \qquad + (4r+1) \sqrt{\frac{\gamma}{\alpha}} \sum_0^\infty (-)^n c_n \begin{Bmatrix} \alpha & \gamma & \\ 2r & n & c_1 + c_2 \end{Bmatrix} = 0. \end{aligned}\right\} . \quad (147)$$

For the third disc, we apply the forward transformation twice. It is not necessary to write down the ensuing value of V on this disc, but the final conditions become

$$\left.\begin{aligned} &c_{2r+1} = 0 \\ &c_0 + \sqrt{\frac{\alpha}{\gamma}} \sum_0^\infty a_n \begin{Bmatrix} \alpha & \gamma \\ n & 0 \end{Bmatrix} c_1 + c_2 \Big\} + \sqrt{\frac{\beta}{\gamma}} \sum_0^\infty b_n \begin{Bmatrix} \beta & \gamma \\ n & 0 \end{Bmatrix} c_2 \Big\} = \frac{2}{\pi} V_3 \\ &c_{2r} + (4r+1) \sqrt{\frac{\alpha}{\gamma}} \sum_0^\infty a_n \begin{Bmatrix} \alpha & \gamma \\ n & 2r \end{Bmatrix} c_1 + c_2 \Big\} + (4r+1) \sqrt{\frac{\beta}{\gamma}} \sum_0^\infty b_n \begin{Bmatrix} \beta & \gamma \\ n & 2r \end{Bmatrix} c_2 \Big\} = 0 \end{aligned}\right\} . \quad (148)$$

The generalisation of these six coefficient relations to any number of discs is clear now by inspection, and they can be used for approximations when the discs are far apart, in regard to coefficients of capacity and induction of such a system. The potential is

$$V = \sum_0^\infty \{a_{2r} P_{2r}(\mu)\, q_{2r}(\zeta') + \ldots + \ldots\}$$

where each of the three terms transforms as usual into a definite integral. In front of the foremost disc at $z = c_2$,

$$V = \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-\lambda z} J_0(\lambda\rho) \frac{d\lambda}{\sqrt{\lambda}} \Big\{ \sum_0^\infty a_{2r} \sqrt{\alpha}\, J_{2r+\frac{1}{2}}(\lambda\alpha)\, e^{-\lambda c_1} + \sum_0^\infty b_{2r} \sqrt{\beta}\, J_{2r+\frac{1}{2}}(\lambda\beta) + \sum_0^\infty c_{2r} \sqrt{\gamma}\, J_{2r+\frac{1}{2}}(\lambda\gamma)\, e^{+\lambda c_2} \Big\}$$

—also admitting an immediate generalisation to any number of such discs. Writing

$$f_1(\lambda\alpha) = \sum_0^\infty a_{2r} J_{2r+\frac{1}{2}}(\lambda\alpha), \qquad f_2(\lambda\beta) = \sum_0^\infty b_{2r} J_{2r+\frac{1}{2}}(\lambda\beta), \qquad f_3(\lambda\gamma) = \sum_0^\infty c_{2r} J_{2r+\frac{1}{2}}(\lambda\gamma) . \quad (149)$$

we obtain

$$V = \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-\lambda z} J_0(\lambda\rho) \frac{d\lambda}{\sqrt{\lambda}} \{\sqrt{\alpha} f_1(\lambda\alpha)\, e^{-\lambda c_1} + \sqrt{\beta} f_2(\lambda\beta) + \sqrt{\gamma} f_3(\lambda\gamma)\, e^{\lambda c_2}\}.$$

We may now return to the six equations among the coefficients, in order to form integral equations. With

$$\begin{Bmatrix} \alpha & \gamma \\ p & q \end{Bmatrix} c \Big\} = \int_0^\infty e^{-cx} J_{p+\frac{1}{2}}(\alpha x)\, J_{q+\frac{1}{2}}(\gamma x) \frac{dx}{x}$$

we find in the usual way,

$$a_{2r} = -\sqrt{\frac{\beta}{\alpha}} \int_0^\infty e^{-c_1 x} \frac{dx}{x} \{(4r+1)\, J_{2r+\frac{1}{2}}(\alpha x)\} \sum_0^\infty b_{2n} J_{2n+\frac{1}{2}}(\beta x) - \sqrt{\frac{\gamma}{\alpha}} \int_0^\infty e^{-(c_1+c_2)x} \frac{dx}{x} \{(4r+1)\, J_{2r+\frac{1}{2}}(\alpha x)\} \sum_0^\infty c_{2n} J_{2n+\frac{1}{2}}(\gamma x) \quad . \quad (150)$$

with an additional $2V_1/\pi$ if $r = 0$. Or

$$a_{2r} = -\sqrt{\frac{\beta}{\alpha}}\int_0^\infty e^{-c_1 x} f_2(\beta x)\frac{dx}{x}\,.\,(4r+1)\,J_{2r+\frac{1}{2}}(\alpha x)$$
$$-\sqrt{\frac{\gamma}{\alpha}}\int_0^\infty e^{-(c_1+c_2)x} f_3(\gamma x)\frac{dx}{x}\,.\,(4r+1)\,(J_{2r+\frac{1}{2}}(\alpha x). \quad . \quad (151)$$

Let y be a new variable independent of x, multiply by $J_{2r+\frac{1}{2}}(\alpha y)$ and sum for all integral values of r.

We obtain the integral equation

$$\frac{f_1(\alpha y)}{\sqrt{y}} - \frac{2V_1}{\pi\sqrt{y}} J_{\frac{1}{2}}(2y) = -\frac{1}{\pi}\sqrt{\frac{\beta}{\alpha}}\int_0^\infty e^{-c_1 x} f_2(\beta x)\frac{dx}{\sqrt{x}} K_\alpha(x, y)$$
$$-\frac{1}{\pi}\sqrt{\frac{\gamma}{\alpha}}\int_0^\infty e^{-(c_1+c_2)x} f_3(\gamma\pi)\frac{dx}{\sqrt{x}} K_\alpha(x, y), \quad . \quad (152)$$

which is the first of three for the determination of the functions f_1, f_2, f_3. The others are evidently

$$\frac{f_2(\beta y)}{\sqrt{y}} - \frac{2V_2}{\pi\sqrt{y}} J_{\frac{1}{2}}(\beta y) = -\frac{1}{\pi}\sqrt{\frac{\alpha}{\beta}}\int_0^\infty e^{-c_1 x} f_1(\alpha x)\frac{dx}{\sqrt{x}} K_\beta(x, y)$$
$$-\frac{1}{\pi}\sqrt{\frac{\gamma}{\beta}}\int_0^\infty e^{-c_2 x} f_3(\gamma x)\frac{dx}{\sqrt{x}} K_\beta(x, y), \quad . \quad (153)$$

$$\frac{f_3(\gamma y)}{\sqrt{y}} - \frac{2V_3}{\pi\sqrt{y}} J_{\frac{1}{2}}(\gamma y) = -\frac{1}{\pi}\sqrt{\frac{\alpha}{\gamma}}\int_0^\infty e^{-(c_1-c_2)x}\frac{dx}{\sqrt{x}} f_1(\alpha x) K_\gamma(x, y)$$
$$-\frac{1}{\pi}\sqrt{\frac{\beta}{\gamma}}\int_0^\infty e^{-c_2 x} f_2(\beta x)\frac{dx}{\sqrt{x}} K_\gamma(x, y), \quad . \quad (154)$$

and the generalisation to n discs is still evident.

We do not pursue the detailed solution of simple cases, which can be effected.

§ 29. *The Value of a Definite Integral.*

If it be desired to pursue the general theory for unequal discs on the lines of our first discussion of the integral equation, a definite integral is of great value.

We may show that

$$\left.\begin{aligned}\int_0^\infty K_a(x, y) K_b(y, t) &= K_b(x, t) \quad \text{if} \quad b < a\\ &= K_a(x, t) \quad \text{if} \quad a < b\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

For the product $K_a(x, y) K_b(y, t)$ consists of four terms of which the first is

$$[\sin a(x+y) \sin b(y+t)]/(x+y)(y+t).$$

The integral of this is

$$\frac{1}{t-x}\int_0^\infty dy \sin a(x+y) \sin b(y+t)\left\{\frac{1}{y+x}-\frac{1}{y+t}\right\}$$
$$=\frac{1}{t-x}\left\{\int_x^\infty \frac{\sin a\lambda}{\lambda} \sin b(\lambda+t-x)\,d\lambda-\int_t^\infty \frac{\sin b\lambda}{\lambda}\sin a(\lambda+x-t)\,d\lambda\right\}.$$

Another term has a change of sign in x and t, and its integral is

$$-\frac{1}{t-x}\left\{\int_{-x}^\infty \frac{\sin a\lambda}{\lambda}\sin b(\lambda-t-x)\,d\lambda+\int_{-t}^\infty \frac{\sin b\lambda}{\lambda}\sin a(\lambda-x+t)\,d\lambda\right\}.$$

The sum of the two integrals becomes

$$\frac{2}{t-x}\sin b(t-x)\int_0^\infty \sin a\lambda\cos b\lambda\frac{d\lambda}{\lambda}-\sin a(x-t)\int_0^\infty \sin b\lambda\cos a\lambda\frac{d\lambda}{\lambda}\bigg\}.$$

Now if $a<b$,

$$\int_0^\infty \sin a\lambda\cos b\lambda\frac{d\lambda}{\lambda}=0,\qquad \int_0^\infty \sin b\lambda\cos a\lambda\frac{d\lambda}{\lambda}=\pi,$$

so that the sum is

$$\pi\frac{\sin b(t+x)}{t+x},$$

where b is the smaller of the quantities (a, b). The other pair of integrals differ only in the sign of x or t, and yield as their sum,

$$\pi\frac{\sin b(t+x)}{t+x},$$

so that the whole integral is $K_b(x, t)$.

This theorem is essentially the same as one given by HARDY.